Linear Transformations

January 31, 2007

The Null Space and the Range of a Linear Transformation

- Let $T : V \to W$ be a linear transformation.

- The **null space** (or **kernel**) $N(T)$ of T is the set of all vectors x in V such that $T(x) = 0$.

- The **range** (or **image**) $R(T)$ of T is the subset of W consisting of all images (under T) of vectors in V.

Theorem. *Let V and W be vector spaces, and let $T : V \to W$ be linear. If $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V, then*

$$R(T) = \textbf{span}(\{T(v_1), T(v_2), \ldots, T(v_n)\}).$$

Nullity and Rank

- Let V and W be vector spaces, and let $T : V \to W$ be linear.
- Suppose that $N(T)$ and $R(T)$ are finite-dimensional.
- The **nullity** of T, denoted nullity(T), is defined to be the dimension of $N(T)$.

Nullity and Rank

- Let V and W be vector spaces, and let $T : V \to W$ be linear.
- Suppose that $N(T)$ and $R(T)$ are finite-dimensional.
- The **nullity** of T, denoted nullity(T), is defined to be the dimension of $N(T)$.
- The **rank** of T, denoted rank(T), is defined to be the dimension of $R(T)$.

Dimension Theorem

Theorem. *Let V and W be vector spaces, and let $T : V \to W$ be linear. If V is finite dimensional, then*

$$\textit{nullity}(T) + \textit{rank}(T) = \textit{dim}(V).$$

Theorem. *Let V and W be vector spaces, and let $T : V \to W$ be lineaar. Then T is one-to-one if and only if $N(T) = \{0\}$.*

Theorem. *Let V and W be vector spaces, and let $T : V \to W$ be lineaar. Then T is one-to-one if and only if $N(T) = \{0\}$.*

Theorem. *Let V and W be vector spaces of equal (finite) dimension, and let $T : V \to W$ be linear. Then the following are equivalent:*

1. *T is one-to-one.*

2. *T is onto.*

3. *rank$(T) =$ dim(V).*

Linear Transformations are Determined by the Values on a Basis

Theorem. *Let* V *and* W *be vector spaces over* F*, and suppose that* $\{v_1, v_2, \ldots, v_n\}$ *is a basis for* V*. For* $w_1, w_2, \ldots, w_n$ *in* W*, there exists exactly one linear transformation* $T : V \to W$ *such that* $T(v_i) = w_i$ *for* $i = 1, 2, \ldots, n$*.*

Corollary. *Let* V *and* W *be vector spaces, and suppose that* V *has a finite basis* $\{v_1, v_2, \ldots, v_n\}$*. If* $U, T : V \to W$ *are linear and* $U(v_i) = T(v_i)$ *for* $i = 1, 2, \ldots, n$*, then* $U = T$*.*