Worksheet for May 14

MATH 24 — SPRING 2014

Sample Solutions

(A) A the harmonics of a vibrating string of length one are describe by the functions

$$\sin(n\pi x), \qquad n = 1, 2, 3, \ldots$$

on the unit interval $[0, 1]$.

An actual vibrating string will typically overlay several harmonic vibrations resulting in more complex patterns that are represented by linear combinations of harmonics:

$$h(x) = a_1 \sin(\pi x) + a_2 \sin(2\pi x) + a_3 \sin(3\pi x) + a_4 \sin(4\pi x) + \cdots$$

The theory of inner product spaces allows us to recover the amplitudes $a_1, a_2, a_3, \ldots$ of the components of the vibration pattern $h(x)$.

1.– Show that the functions $\sin(\pi x), \sin(2\pi x), \sin(3\pi x), \ldots$ form an orthogonal family with respect to the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$ on the space of continuous functions on the unit interval $[0, 1]$.

Solution — Using the identity $2\sin(\alpha)\sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta)$, we get

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = \frac{1}{2}\int_0^1 \cos((n-m)\pi x)\,dx - \frac{1}{2}\int_0^1 \cos((m+n)\pi x).$$

If k is a nonzero integer, then

$$\int_0^1 \cos(k\pi x)\,dx = \frac{1}{k\pi}\sin(k\pi 1) - \frac{1}{k\pi}\sin(k\pi 0) = 0.$$

So if $m \neq n$ then

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = 0.$$

However, then

$$\int_0^1 \sin^2(n\pi x)\,dx = \frac{1}{2}\int_0^1 dx = \frac{1}{2}.$$

So the functions $\sin(n\pi x)$ are orthogonal to each other with respect to the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$. So they do form an orthogonal family, but since they are not unit vectors they do not form an orthonormal family.

2.– Explain how to use this to recover the amplitudes $a_1, a_2, a_3, \ldots$ of a given vibration pattern $h(x)$.

Solution — Knowing that

$$h(x) = a_1 \sin(\pi x) + a_2 \sin(2\pi x) + a_3 \sin(3\pi x) + a_4 \sin(4\pi x) + \cdots$$

we see that

$$\begin{aligned}\langle h(x), \sin(\pi x)\rangle &= a_1\langle \sin(\pi x), \sin(\pi x)\rangle + a_2\langle \sin(2\pi x), \sin(\pi x)\rangle + a_3\langle \sin(3\pi x), \sin(\pi x)\rangle + \cdots \\ &= a_1\langle \sin(\pi x), \sin(\pi x)\rangle = \frac{a_1}{2}.\end{aligned}$$

Similarly,

$$\langle h(x), \sin(2\pi x)\rangle = \frac{a_2}{2}, \quad \langle h(x), \sin(3\pi x)\rangle = \frac{a_3}{2}, \quad \ldots$$

In general, the formula

$$a_n = 2\langle h(x), \sin(n\pi x)\rangle = 2\int_0^1 h(x)\sin(n\pi x)\,dx$$

allows us to recover the coefficients $a_1, a_2, a_3, \ldots$

(B) The Legendre Polynomials are a family of polynomials $P_0(x), P_1(x), P_2(x), \ldots$ with the following three properties:

(i) $\deg(P_n(x)) = n$,

(ii) $P_n(1) = 1$,

(iii) $\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0$ when $m \neq n$.

These three properties uniquely determine these polynomials! For example the first two properties dictate that $P_0(x)$ is the constant polynomial 1. For $P_1(x)$, the first two properties dictate that this is a linear function through the point $(1, 1)$. The third property entails that $\int_{-1}^{1} P_1(x)\,dx = 0$; the only possibility is $P_1(x) = x$.

1.– Show that $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(x)g(x)\,dx$ is an inner product on $\mathsf{P}(\mathbb{R})$, the vector space of all polynomials with coefficients in $\mathbb{R}$.

Solution — Since $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(x)g(x)\,dx$ defines an inner product on the space of continuous functions on the interval $[-1, 1]$ and polynomials define continuous functions, we see that $\langle \bullet, \bullet\rangle$ is symmetric bilinear and $\langle f(x), f(x)\rangle \geq 0$ for all $f(x) \in \mathsf{P}(\mathbb{R})$. It is not immediately clear that $\langle f(x), f(x)\rangle = 0$ implies that $f(x)$ is the zero polynomials since all we know is that if $\langle f(x), f(x)\rangle = 0$ then $f(x) = 0$ for all $x \in [-1, 1]$. However, since $[-1, 1]$ is infinite, this does tell us that $f(x)$ has more than $\deg(f(x))$ roots. Since a nonzero polynomial has no more than $\deg(f(x))$ roots, we conclude that $\langle f(x), f(x)\rangle = 0$ does imply that $f(x)$ is the zero polynomial.

2.– Show that $P_n(x)$ must be a scalar multiple of

$$x^n - \frac{\langle x^n, P_{n-1}(x)\rangle}{\langle P_{n-1}(x), P_{n-1}(x)\rangle} P_{n-1}(x) - \cdots - \frac{\langle x^n, P_1(x)\rangle}{\langle P_1(x), P_1(x)\rangle} P_1(x) - \frac{\langle x^n, P_0(x)\rangle}{\langle P_0(x), P_0(x)\rangle} P_0(x).$$

Solution — Because of property (i), we know that $\{P_0(x), P_1(x), \ldots, P_n(x)\}$ forms a basis for $\mathsf{P}_n(\mathbb{R})$, so we know that there are unique scalars $a_0, a_1, \ldots, a_n$ such that

$$x^n = a_0 P_0(x) + a_1 P_1(x) + \cdots a_n P_n(x).$$

By the orthogonality relations (iii), we see that

$$\langle x^n, P_0(x)\rangle = a_0 \langle P_0(x), P_0(x)\rangle, \quad \langle x^n, P_1(x)\rangle = a_1 \langle P_1(x), P_1(x)\rangle, \quad \ldots$$

Therefore,

$$\begin{aligned} &x^n - \frac{\langle x^n, P_{n-1}(x)\rangle}{\langle P_{n-1}(x), P_{n-1}(x)\rangle} P_{n-1}(x) - \cdots - \frac{\langle x^n, P_1(x)\rangle}{\langle P_1(x), P_1(x)\rangle} P_1(x) - \frac{\langle x^n, P_0(x)\rangle}{\langle P_0(x), P_0(x)\rangle} P_0(x) \\ &\quad = x^n - a_{n-1} P_{n-1}(x) - \cdots - a_1 P_1(x) - a_0 P_0(x) = a_n P_n(x). \end{aligned}$$

Since $\deg(x^n) = n$, we cannot have $a_n = 0$ and hence $P_n(x)$ is $1/a_n$ times the given polynomial.

3.– Use the above to compute $P_2(x), P_3(x), P_4(x), P_5(x)$.

Solution — We know that $P_2(x)$ is a scalar multiple of

$$x^2 - \frac{\langle x^2, P_1(x)\rangle}{\langle P_1(x), P_1(x)\rangle} P_1(x) - \frac{\langle x^2, P_0(x)\rangle}{\langle P_0(x), P_0(x)\rangle} P_0(x) = x^2 - \frac{1}{3}.$$

Since we need $P_2(1) = 1$ per (ii), we see that

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}.$$

Similarly, we know that $P_3(x)$ is a scalar multiple of

$$x^3 - \frac{2}{5}x.$$

Therefore,

$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x.$$

Repeating this process, we find that

$$\begin{aligned} P_4(x) &= \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}, \\ P_5(x) &= \frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x. \end{aligned}$$