Slides for May 28

Math 24 — Spring 2014

Nilpotent Transformations

Theorem

Suppose $T : \mathrm{V} \to \mathrm{V}$ *is a nilpotent linear operator on a finite dimensional vector space. There are vectors* $w_1, w_2, \ldots, w_\ell$ *such that*

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_\ell$$

where each $W_i = \mathrm{span}\{w_i, T(w_i), T^2(w_i), \ldots\}$ *is the* T*-cyclic space generated by* w_i*.*

Jordan basis β:

$\bullet T^3(w_1)$	$\bullet T^2(w_2)$	$\bullet T^2(w_3)$	$\bullet w_4$
$\bullet T^2(w_1)$	$\bullet T(w_2)$	$\bullet T(w_3)$	
$\bullet T(w_1)$	$\bullet w_2$	$\bullet w_3$	
$\bullet w_1$			

Nilpotent Transformations

$$[T]_\beta = \left(\begin{array}{cccc|ccc|ccc|c}
\mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0}
\end{array}\right)$$

Nilpotent Transformations

Jordan basis β:

$\bullet T^3(w_1)$	$\bullet T^2(w_2)$	$\bullet T^2(w_3)$	$\bullet w_4$
$\bullet T^2(w_1)$	$\bullet T(w_2)$	$\bullet T(w_3)$	
$\bullet T(w_1)$	$\bullet w_2$	$\bullet w_3$	
$\bullet w_1$			

$\text{nullity}(T) = 4$ (total number of cycles)

(dots in the first row)

$\text{nullity}(T^2) = 7$ (dots in first two rows)

$\text{nullity}(T^3) = 10$ (dots in first three rows)

$\text{nullity}(T^4) = 11$ (total dots)

(dimension of space)

Jordan Canonical Form

Theorem

Suppose $T : \mathsf{V} \to \mathsf{V}$ *is a linear operator on a finite dimensional vector space* V*. Suppose the characteristic polynomial of* T *splits*

$$\det(T - tI) = (\lambda_1 - t)^{m_1}(\lambda_2 - t)^{m_2} \cdots (\lambda_p - t)^{m_p},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_p$ *are distinct. Then, for each* $i = 1, 2, \ldots, p$:

- $\mathsf{K}_{\lambda_i} = \mathsf{N}((T - \lambda_i I)^{m_i})$
- $\dim(\mathsf{K}_{\lambda_i}) = m_i$
- *The restriction of* $T - \lambda_i I$ *to* K_i *is nilpotent.*

Furthermore,

$$\mathsf{V} = \mathsf{K}_{\lambda_1} \oplus \mathsf{K}_{\lambda_2} \oplus \cdots \oplus \mathsf{K}_{\lambda_p}$$

and if β_i *is a Jordan basis for* $T - \lambda_i I$ *restricted to* K_{λ_i}*, then* $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_p$ *is a Jordan basis for* T*.*

Jordan Canonical Form

To find a Jordan basis for $T : \mathsf{V} \to \mathsf{V} \ldots$
For each eigenvalue λ of T:

1. Compute

$$\mathsf{N}(T - \lambda I) \subseteq \mathsf{N}(T - \lambda I)^2 \subseteq \mathsf{N}(T - \lambda I)^3 \subseteq \cdots$$

   until you reach K_λ.

2. Write the dot pattern that matches the nullities you just found.
3. Fill in the dot pattern by finding appropriate initial vectors for each cycle, starting with the longest cycles.
4. Write down the base for K_λ, cycle per cycle, each ending with the initial vector.

Assemble the bases you just found into a Jordan basis for T.