Slides for May 19

Math 24 — Spring 2014

Eigenvectors of Normal Operators

Theorem

Suppose V *is a finite dimensional inner product space. If* $T : \mathrm{V} \to \mathrm{V}$ *is a normal operator then:*

(a) $\|T(x)\| = \|T^*(x)\|$ *for all* $x \in \mathrm{V}$.

(b) $T(x) = \lambda x$ *iff* $T^*(x) = \overline{\lambda} x$.

(c) *If* $T(x_1) = \lambda_1 x_1$, $T(x_2) = \lambda_2 x_2$ *and* $\lambda_1 \neq \lambda_2$ *then* $\langle x_1, x_2 \rangle = 0$.

Proof of (a).

On the one hand, $\|T(x)\|^2 = \langle T(x), T(x) \rangle = \langle x, T^* T(x) \rangle$.
On the other hand, $\|T^*(x)\|^2 = \langle T^*(x), T^*(x) \rangle = \langle x, TT^*(x) \rangle$ because $T^{**} = T$.
Since $T^* T = TT^*$, we see that $\|T(x)\|^2 = \|T^*(x)\|^2$. □

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Proof of (b).

Since $T - \lambda I$ is also normal with adjoint $T^* - \overline{\lambda} I$, it follows from (a) that $\mathrm{N}(T - \lambda I) = \mathrm{N}(T^* - \overline{\lambda} I)$. □

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Proof of (c).

On the one hand, $\langle T(x_1), x_2 \rangle = \langle \lambda_1 x_1, x_2 \rangle = \lambda_1 \langle x_1, x_2 \rangle$.
On the other hand, $\langle x_1, T^*(x_2) \rangle = \langle x_1, \overline{\lambda_2} x_2 \rangle = \lambda_2 \langle x_1, x_2 \rangle$ because of (b).
Since $\langle T(x_1), x_2 \rangle = \langle x_1, T^*(x_2) \rangle$, we see that $(\lambda_1 - \lambda_2) \langle x_1, x_2 \rangle = 0$.
If $\lambda_1 \neq \lambda_2$ then $\langle x_1, x_2 \rangle = 0$. □

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(b) $T(x) = \lambda x$ *iff* $T^*(x) = \overline{\lambda} x$.

(c) *If* $T(x_1) = \lambda_1 x_1$, $T(x_2) = \lambda_2 x_2$ *and* $\lambda_1 \neq \lambda_2$ *then* $\langle x_1, x_2 \rangle = 0$.

Corollary

For a normal operator $T : \mathrm{V} \to \mathrm{V}$ *on a finite dimensional inner product space:*

- *The eigenvalues of* T *are precisely the conjugates of the eigenvalues of* T^*.
- *The eigenspaces of* T *and* T^* *corresponding to conjugate eigenvalues are the same.*
- *The eigenspaces of* T *are mutually orthogonal to each other.*

Diagonalization of Normal Operators

Theorem

Suppose V *is a finite dimensional complex inner product space. A linear operator* $T : \mathsf{V} \to \mathsf{V}$ *is normal if and only if there is an orthonormal basis of eigenvectors of* T*.*

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T and let $\mathsf{E}_1, \mathsf{E}_2, \ldots, \mathsf{E}_k$ be the corresponding eigenspaces.
Two key observations:

- T is diagonalizable if and only if $\mathsf{E}_1 + \mathsf{E}_2 + \cdots + \mathsf{E}_k = \mathsf{V}$.
- Since the eigenspaces $\mathsf{E}_1, \mathsf{E}_2, \ldots, \mathsf{E}_k$ are mutually orthogonal, T is diagonalizable if and only if it has an orthonormal basis of eigenvectors.

So it is enough to show that $\mathsf{E}_1 + \mathsf{E}_2 + \cdots + \mathsf{E}_k = \mathsf{V}$.

Diagonalization of Normal Operators

Lemma (A)

Each orthogonal complement $\mathsf{E}_i^{\perp}$ *is* T*-invariant.*

Proof of Lemma (A).

First note that $\mathsf{E}_i = \mathsf{N}(T^* - \overline{\lambda_i} I)$ by part (b) of the first Theorem. Suppose $y \in \mathsf{E}_i^{\perp}$. Given any $x \in \mathsf{E}_i$, we have

$$\langle T(y), x \rangle = \langle y, T^*(x) \rangle = \langle y, \overline{\lambda} x \rangle = \lambda \langle y, x \rangle = 0.$$

Therefore, $T(y) \in \mathsf{E}_i^{\perp}$ too. □

Diagonalization of Normal Operators

Lemma (A)

Each orthogonal complement $\mathsf{E}_i^\perp$ *is* T*-invariant.*

Lemma (B)

The space $\mathsf{W} = (\mathsf{E}_1 + \mathsf{E}_2 + \cdots + \mathsf{E}_k)^\perp = \mathsf{E}_1^\perp \cap \mathsf{E}_2^\perp \cap \cdots \cap \mathsf{E}_k^\perp$ *is* T*-invariant.*

Proof of Lemma (B).

If $x \in \mathsf{W}$ then $x \in \mathsf{E}_i^\perp$, so $T(x) \in \mathsf{E}_i^\perp$ by Lemma (A). Since this is true for $i = 1, 2, \ldots, k$, we see that $T(x) \in \mathsf{W}$. $\square$

Diagonalization of Normal Operators

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Each orthogonal complement $\mathsf{E}_i^\perp$ *is* T*-invariant.*

Lemma (B)

The space $\mathsf{W} = (\mathsf{E}_1 + \mathsf{E}_2 + \cdots + \mathsf{E}_k)^\perp = \mathsf{E}_1^\perp \cap \mathsf{E}_2^\perp \cap \cdots \cap \mathsf{E}_k^\perp$ *is* T*-invariant.*

Lemma (C)

$\mathsf{W} = \{0\}$ *and therefore* $\mathsf{E}_1 + \mathsf{E}_2 + \cdots + \mathsf{E}_k = \{0\}^\perp = \mathsf{V}$.

Proof of Lemma (C).

The restriction T_W has no eigenvectors nor eigenvalues, so the characterstic polynomial of T_W must be constant otherwise it would have a root in $\mathbb{C}$. Since the degree of the characteristic polynomial is $\dim(\mathsf{W})$, it must be that $\mathsf{W} = \{0\}$. □

Diagonalization of Self-Adjoint Operators

Theorem

Suppose V *is a finite dimensional complex inner product space. A linear operator* $T : \mathsf{V} \to \mathsf{V}$ *is self-adjoint if and only if all eigenvalues of* T *are real and there is an orthonormal basis of eigenvectors of* T*.*

Corollary

The eigenvalues of a real symmetric matrix are all real. Furthermore, every real symmetric matrix is diagonalizable.

Theorem

Suppose V *is a finite dimensional real inner product space. A linear operator* $T : \mathsf{V} \to \mathsf{V}$ *is self-adjoint if and only if there is an orthonormal basis of eigenvectors of* T*.*