Slides for May 16

MATH 24 — SPRING 2014

Dual Transformations

Theorem

If $T : \mathrm{V} \to \mathrm{W}$ *is a linear operator, then* $T^t : \mathrm{W}^* \to \mathrm{V}^*$ *is a linear operator too where*

$$T^t(f)(\bullet) = f(T\bullet)$$

for all $f \in \mathrm{W}^*$.

Given $f, g \in \mathrm{W}^*$, we have

$$T^t(f+g)(\bullet) = (f+g)(T\bullet) = f(T\bullet)+g(T\bullet) = T^t(f)(\bullet)+T^t(g)(\bullet).$$

If c is a scalar, we have

$$T^t(cf)(\bullet) = (cf)(T\bullet) = c(f(T\bullet)) = cT^t(f)(\bullet).$$

Self-Duality of Inner Product Spaces

Theorem

Let V *be a finite dimensional inner product space. There is a conjugate-linear isomorphism* $\theta : \mathrm{V}^* \to \mathrm{V}$ *such that*

$$f(\bullet) = \langle \bullet, \theta(f) \rangle$$

for all $f \in \mathrm{V}^*$.

If $\{v_1, v_2, \ldots, v_n\}$ is any orthonormal basis for V, then

$$\theta(f) = \overline{f(v_1)}v_1 + \overline{f(v_2)}v_2 + \cdots + \overline{f(v_n)}v_n.$$

The inverse $\theta^{-1} : \mathrm{V} \to \mathrm{V}^*$ is given by

$$\theta^{-1}(w)(\bullet) = \langle \bullet, w \rangle$$

for all $w \in \mathrm{V}$.

Adjoint Transformation

Theorem

Let V *be a finite dimensional inner product space. For every linear operator* $T : \mathrm{V} \to \mathrm{V}$ *there is a unique linear operator* $T^* : \mathrm{V} \to \mathrm{V}$ *such that*

$$\langle Tx, y \rangle = \langle x, T^* y \rangle$$

for all $x, y \in \mathrm{V}$.

The adjoint $T^* : \mathrm{V} \to \mathrm{V}$ is $T^* = \theta T^t \theta^{-1}$ because

$$\langle Tx, y \rangle = \theta^{-1}(y)(Tx) = T^t \theta^{-1}(y)(x) = \langle x, \theta T^t \theta^{-1}(y) \rangle$$

for all $x, y \in \mathrm{V}$.

T^* is called the **adjoint** of T

Adjoint Matrix

Recall that the adjoint A^* of a matrix $A \in \mathrm{M}_{n\times n}(\mathbb{C})$ is the conjugate transpose of A.

Theorem

Let V *be a finite dimensional inner product space with orthonormal basis* $\beta = \{v_1, v_2, \ldots, v_n\}$*. For every linear operator* $T : \mathrm{V} \to \mathrm{V}$ *we have* $[T^*]_\beta = [T]_\beta^*$.

Because

$$\langle T^* v_i, v_j\rangle = \overline{\langle v_j, T^* v_i\rangle} = \overline{\langle T v_j, v_i\rangle},$$

the (j, i)-th entry of $[T^*]_\beta$ is the conjugate of the (i, j)-th entry of $[T]_\beta$.

Normal Operators

Definition

A linear operator $T : \mathsf{V} \to \mathsf{V}$ is **normal** if it commutes with its adjoint: $T^*T = TT^*$.

Theorem

Suppose $T : \mathsf{V} \to \mathsf{V}$ is a linear operator on a finite dimensional complex inner product space V. Then T is normal if and only if T has an orthonormal basis of eigenvectors.

The 'only if' part will be shown next week...

Self-Adjoint Operators

Definition

A linear operator $T : \mathrm{V} \to \mathrm{V}$ is **self-adjoint** if it equals its adjoint: $T = T^*$.

Theorem

Suppose $T : \mathrm{V} \to \mathrm{V}$ is a linear operator on a finite dimensional real inner product space V. Then T is self-adjoint if and only if T has an orthonormal basis of eigenvectors.

The 'only if' part will be shown next week...

Orthogonal Complements

Let V be an inner product space.
The **orthogonal complement** of a subspace W of V is

$$\mathsf{W}^\perp = \{x \in \mathsf{V} : \langle x, y \rangle = 0 \text{ for all } y \in \mathsf{W}\}.$$

If $\mathsf{W} = \mathrm{span}(S)$ then

$$\mathsf{W}^\perp = \{x \in \mathsf{V} : \langle x, y \rangle = 0 \text{ for all } y \in S\}.$$

Properties

- $\mathsf{W}^\perp$ is a subspace of V.
- $\mathsf{W} \cap \mathsf{W}^\perp = \{0\}$.
- $(\mathsf{W}^\perp)^\perp = \mathsf{W}$.
- $\mathsf{V}^\perp = \{0\}$ and $\{0\}^\perp = \mathsf{V}$.

Orthogonal Projections

Theorem

If V *is a finite dimensional inner product space and* W *is a subspace of* V, *then* $V = W \oplus W^{\perp}$ *(direct sum).*
In other words, every $x \in V$ *has a unique decomposition* $x = u + v$ *where* $u \in W$ *and* $v \in W^{\perp}$.

If $\{w_1, w_2, \ldots, w_k\}$ is any orthonormal basis for the subspace W, then the component u in the above decomposition must be

$$P(x) = \langle x, w_1 \rangle w_1 + \langle x, w_2 \rangle w_2 + \cdots + \langle x, w_k \rangle w_k.$$

The linear transformation $P : V \to V$ so defined is the **orthogonal projection** onto W.