Slides for May 12

Math 24 — Spring 2014

Inner Products over $\mathbb{R}$

Definition

An **inner product** on a vector space V over $\mathbb{R}$ is a positive definite symmetric bilinear form $\langle \bullet, \bullet \rangle : V \times V \to \mathbb{R}$:

- **Symmetry:** $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in V$.
- **Bilinearity:** For all $x, x', y, y' \in V$ and all scalars c:

$$\langle x + x', y \rangle = \langle x, y \rangle + \langle x', y \rangle, \quad \langle cx, y \rangle = c\langle x, y \rangle;$$
$$\langle x, y + y' \rangle = \langle x, y \rangle + \langle x, y' \rangle, \quad \langle x, cy \rangle = c\langle x, y \rangle.$$

- **Positive Definiteness:** For every $x \in V$, $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ only when $x = 0$.

Example. The **standard inner product** on $\mathbb{R}^n$ is

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Inner Products over $\mathbb{C}$

Definition

An **inner product** on a vector space V over $\mathbb{C}$ is a positive definite conjugate-symmetric sesquilinear form $\langle \bullet, \bullet \rangle : V \times V \to \mathbb{C}$:

- **Conjugate-Symmetry:** $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in V$.
- **Sesquilinearity:** For all $x, x', y, y' \in V$ and all scalars c:

$$\langle x + x', y \rangle = \langle x, y \rangle + \langle x', y \rangle, \quad \langle cx, y \rangle = c\langle x, y \rangle;$$
$$\langle x, y + y' \rangle = \langle x, y \rangle + \langle x, y' \rangle, \quad \langle x, cy \rangle = \overline{c}\langle x, y \rangle.$$

- **Positive Definiteness:** For every $x \in V$, $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ only when $x = 0$.

Example. The **standard inner product** on $\mathbb{C}^n$ is

$$\langle x, y \rangle = x_1\overline{y_1} + x_2\overline{y_2} + \cdots + x_n\overline{y_n}.$$

Spaces of continuous functions

Example

The space $C([a,b])$ of continuous real-valued functions on the interval $[a,b]$ is a vector space over $\mathbb{R}$ with inner product

$$\langle f, g \rangle = \int_a^b f(t)g(t)\,dt.$$

The tricky thing to check is positive definiteness:

- Since $f(t)^2 \geq 0$, we certainly have $\langle f, f \rangle \geq 0$.
- If $f(t_0) \neq 0$ then there is a small interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ over which $f(t)^2 \geq f(t_0)^2/2$. Thus $\langle f, f \rangle \geq \varepsilon f(t_0)^2 > 0$.

The Frobenius inner product

Definition

The **adjoint** of a matrix $A \in \mathsf{M}_{n\times n}(\mathbb{C})$ is the matrix

$$A^* = \overline{A^t} = \begin{pmatrix} \overline{A_{11}} & \overline{A_{21}} & \cdots & \overline{A_{n1}} \\ \overline{A_{12}} & \overline{A_{22}} & \cdots & \overline{A_{n2}} \\ \vdots & \vdots & & \vdots \\ \overline{A_{1n}} & \overline{A_{2n}} & \cdots & \overline{A_{nn}} \end{pmatrix}.$$

Example

For $A, B \in \mathsf{M}_{n\times n}(\mathbb{C})$,

$$\langle A, B \rangle = \mathrm{tr}(B^* A)$$

defines an inner product on $\mathsf{M}_{n\times n}(\mathbb{C})$.

Norms

Definition

The **norm** of a vector x in an inner product space V is

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Theorem

Suppose V *is an inner product space.*

- $\|x\| \geq 0$ *for every* $x \in \mathrm{V}$ *and* $\|x\| = 0$ *only when* $x = 0$.
- $\|cx\| = |c|\|x\|$ *for every* $x \in \mathrm{V}$ *and every scalar* c.
- **Triangle Inequality:** $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in \mathrm{V}$.
- **Cauchy–Schwarz:** $|\langle x, y \rangle| \leq \|x\|\|y\|$ *for all* $x, y \in \mathrm{V}$.

The Cauchy–Schwarz Inequality for $\mathbb{R}$

Given $x, y \in \mathrm{V}$ with $y \neq 0$, let

$$q(t) = \langle x - ty, x - ty \rangle = \langle x, x \rangle - 2\langle x, y \rangle t + \langle y, y \rangle t^2.$$

The minimum of of this quadratic occurs when $t = \langle x, y \rangle / \langle y, y \rangle$, and the nonnegative value is

$$q\left(\frac{\langle x, y \rangle}{\langle y, y \rangle}\right) = \langle x, x \rangle - \frac{\langle x, y \rangle^2}{\langle y, y \rangle}.$$

Multiplying through by $\langle y, y \rangle \geq 0$, we obtain that

$$0 \leq \langle x, x \rangle \langle y, y \rangle - \langle x, y \rangle^2$$

or, equivalently,

$$\langle x, y \rangle^2 \leq \langle x, x \rangle \langle y, y \rangle = \|x\|^2 \|y\|^2.$$

Therefore, $|\langle x, y \rangle| = \sqrt{\langle x, y \rangle^2} \leq \sqrt{\|x\|^2 \|y\|^2} = \|x\| \|y\|$.

Triangle and Cauchy–Schwarz Inequalities for $\mathbb{R}$

On the one hand,

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &= \|x\|^2 + 2\langle x,y\rangle + \|y\|^2.\end{aligned}$$

On the other hand,

$$(\|x\| + \|y\|)^2 = \|x\|^2 + 2\|x\|\|y\| + \|y\|^2.$$

Therefore,

$$\|x+y\|^2 \le (\|x\| + \|y\|)^2$$

if and only if

$$\langle x,y\rangle \le \|x\|\|y\|.$$

Orthogonality

Definition

Let V be an inner product space.

- A **unit vector** is a vector of norm 1.
- Two vectors $x, y \in \mathrm{V}$ are **orthogonal** if $\langle x, y \rangle = 0$.

Definition

An **orthogonal basis** for an n-dimensional inner product space V is a basis $\{v_1, v_2, \ldots, v_n\}$ such that $\langle v_i, v_j \rangle = 0$ when $i \neq j$.
If, moreover, $\langle v_i, v_i \rangle = 1$ for each i, we say that $\{v_1, v_2, \ldots, v_n\}$ is an **orthonormal basis**.

The standard basis for $\mathbb{R}^n$ and $\mathbb{C}^n$ is orthonormal for the standard inner product.

Othonormal Bases

Theorem

Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ *is an orthonormal basis for the inner product space* V*. The coordinates of a vector* $x \in \mathrm{V}$ *are*

$$[x]_\beta = (\langle x, v_1\rangle, \langle x, v_2\rangle, \ldots, \langle x, v_n\rangle).$$

Proof.

Suppose

$$x = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n,$$

so $[x]_\beta = (a_1, a_2, \ldots, a_n)$. Then

$$\langle x, v_i\rangle = a_1\langle v_1, v_i\rangle + a_2\langle v_2, v_i\rangle + \cdots + a_n\langle v_n, v_i\rangle.$$

Since $\langle v_j, v_i\rangle = 0$ when $i \neq j$ and $\langle v_i, v_i\rangle = 1$, we see that $\langle x, v_i\rangle = a_i$. □

Gram–Schmidt Algorithm

Suppose $\{x_1, x_2, \ldots, x_n\}$ is a basis for an inner product space V. Then the vectors

$$\begin{aligned}
v_1 &= x_1, \\
v_2 &= x_2 - \frac{\langle x_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1, \\
v_3 &= x_3 - \frac{\langle x_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 - \frac{\langle x_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1, \\
\vdots &\quad \vdots \\
v_n &= x_n - \frac{\langle x_n, v_{n-1} \rangle}{\langle v_{n-1}, v_{n-1} \rangle} v_{n-1} - \cdots - \frac{\langle x_n, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1,
\end{aligned}$$

form an orthogonal basis for V.