Slides for April 30

MATH 24 — SPRING 2014

Permutations

Definition

An n-**permutation** is a rearrangement $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ of the numbers $1, 2, \ldots, n$.

There are exactly six 3-permutations:

$$\begin{array}{ccc} (1,2,3) & (3,1,2) & (2,3,1) \\ (1,3,2) & (3,2,1) & (2,1,3) \end{array}$$

An **inversion** in an n-permutation is a pair of numbers $p < q$ such that $\sigma_p > \sigma_q$.

$$\begin{array}{ccc} \varnothing & \{(1,2),(1,3)\} & \{(1,3),(2,3)\} \\ \{(2,3)\} & \{(1,2),(1,3),(2,3)\} & \{(1,2)\} \end{array}$$

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The **sign** of an n-permutation is $\text{sign}(\sigma) = (-1)^P$ where P is the number of inversions in σ.

$$\begin{array}{ccc} \text{sign}(1,2,3) = 1 & \text{sign}(3,1,2) = 1 & \text{sign}(2,3,1) = 1 \\ \text{sign}(1,3,2) = -1 & \text{sign}(3,2,1) = -1 & \text{sign}(2,1,3) = -1 \end{array}$$

Determinants

Alternate Definition

The **determinant** of an $n \times n$ matrix A is

$$\det(A) = \sum_{\sigma} \text{sign}(\sigma) A_{1,\sigma_1} A_{2,\sigma_2} \cdots A_{n,\sigma_n},$$

where the summation is over all n-permutations.

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{array}{c} a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} \\ -a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} \end{array}$$

The determinant of an $n \times n$ matrix has $n!$ terms.

Cofactor Expansion

Definition

If A is an $n \times n$ matrix then $\widetilde{A}_{ij}$ denotes the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row and the j-th column of A. The (i,j)-**cofactor** of A is $(-1)^{i+j} \det\left(\widetilde{A}_{ij}\right)$.

Theorem

If A is an $n \times n$ matrix then, for any $1 \leq i \leq n$,

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det\left(\widetilde{A}_{ij}\right).$$

The textbook uses this formula with $i = 1$ to define determinants.

Cofactor Expansion

The cofactor expansion formula can be obtained by looking at the terms

$$\text{sign}(\sigma)A_{1\sigma_1}\cdots A_{i\sigma_i}\cdots A_{n\sigma_n}.$$

If $\widetilde{\sigma}$ is obtained from the n-permutation σ by deleting σ_n and replacing all numbers $\sigma_k > \sigma_n$ by $\sigma_k - 1$, the result $\widetilde{\sigma}$ is an $(n-1)$-permutation.

Example: For the six 3-permutations, we obtain:

- If $\sigma = (1,2,3)$ then $\sigma_3 = 3$ and $\widetilde{\sigma} = (1,2)$.
- If $\sigma = (3,1,2)$ then $\sigma_3 = 2$ and $\widetilde{\sigma} = (2,1)$.
- If $\sigma = (2,3,1)$ then $\sigma_3 = 1$ and $\widetilde{\sigma} = (1,2)$.
- If $\sigma = (1,3,2)$ then $\sigma_3 = 2$ and $\widetilde{\sigma} = (1,2)$.
- If $\sigma = (3,2,1)$ then $\sigma_3 = 1$ and $\widetilde{\sigma} = (2,1)$.
- If $\sigma = (2,1,3)$ then $\sigma_3 = 3$ and $\widetilde{\sigma} = (2,1)$.

Cofactor Expansion

The cofactor expansion formula can be obtained by looking at the terms

$$\text{sign}(\sigma)A_{1\sigma_1}\cdots A_{i\sigma_i}\cdots A_{n\sigma_n}.$$

If $\widetilde{\sigma}$ is obtained from the n-permutation σ by deleting σ_n and replacing all numbers $\sigma_k > \sigma_n$ by $\sigma_k - 1$, the result $\widetilde{\sigma}$ is an $(n-1)$-permutation.

Claim: If $\sigma_n = j$, then $\text{sign}(\sigma) = (-1)^{n+j}\,\text{sign}(\widetilde{\sigma})$.

If we factor out $(-1)^{n+j}A_{nj} = (-1)^{n+j}A_{n\sigma_n}$ from the term above, we obtain

$$\text{sign}(\widetilde{\sigma})A_{1\sigma_1}\cdots A_{n-1,\sigma_{n-1}}.$$

This is precisely the term of $\det(\widetilde{A}_{nj})$ corresponding to $\widetilde{\sigma}$. Therefore, $(-1)^{n+j}\det(\widetilde{A}_{nj})$ is the sum of all terms of $\det(A)$ corresponding to n-permutations with $\sigma_n = j$. Adding all these terms for $j = 1, 2, \ldots, n$, we obtain the cofactor expansion.

Row Exchanges

Theorem

If the $n \times n$ matrix A is obtained from the $n \times n$ matrix B by exchanging rows i and j then $\det(A) = -\det(B)$.

Given an n-permutation σ, let τ be the n-permutation resulting from exchanging σ_i with σ_j in σ.

Claim: $\operatorname{sign}(\sigma) = -\operatorname{sign}(\tau)$.

Therefore the term

$$\operatorname{sign}(\sigma) A_{1\sigma_1} \cdots A_{n\sigma_n}$$

of $\det(A)$ is the negative of the term

$$\operatorname{sign}(\tau) B_{1\tau_1} \cdots B_{n\tau_n}$$

of $\det(B)$.

Multilinearity

Given vectors $v_1, v_2, \ldots, v_n \in F^n$, write

$$\det(v_1, v_2, \ldots, v_n)$$

for the determinant of the $(n \times n)$-matrix whose rows are $v_1, v_2, \ldots, v_n$.

Theorem

(a) $\det(v_1, \ldots, cv_i, \ldots, v_n) = c\det(v_1, \ldots, v_i, \ldots, v_n)$

(b) $\det(v_1, \ldots, v_i + v_i', \ldots, v_n) =$
$\det(v_1, \ldots, v_i, \ldots, v_n) + \det(v_1, \ldots, v_i', \ldots, v_n)$

Note that this only works one row at a time!

Multilinearity

Theorem

(a) $\det(v_1, \ldots, cv_i, \ldots, v_n) = c \det(v_1, \ldots, v_i, \ldots, v_n)$

(b) $\det(v_1, \ldots, v_i + v_i', \ldots, v_n) =$
$\det(v_1, \ldots, v_i, \ldots, v_n) + \det(v_1, \ldots, v_i', \ldots, v_n)$

For (a), suppose A has rows $v_1, \ldots, cv_i, \ldots, v_n$ and B has rows $v_1, \ldots, v_i, \ldots, v_n$.
Given an n-permutation σ, we have:

$$A_{k\sigma_k} = \begin{cases} B_{k\sigma_k} & \text{when } k \neq i, \\ cB_{i\sigma_i} & \text{when } k = i. \end{cases}.$$

Therefore

$$\operatorname{sign}(\sigma) A_{1\sigma_1} \cdots A_{n\sigma_n} = c \operatorname{sign}(\sigma) B_{1\sigma_1} \cdots B_{n\sigma_n}.$$

Multilinearity

Theorem

(a) $\det(v_1, \ldots, cv_i, \ldots, v_n) = c\det(v_1, \ldots, v_i, \ldots, v_n)$

(b) $\det(v_1, \ldots, v_i + v_i', \ldots, v_n) =$
$\det(v_1, \ldots, v_i, \ldots, v_n) + \det(v_1, \ldots, v_i', \ldots, v_n)$

For (b), suppose A has rows $v_1, \ldots, v_i + v_i', \ldots, v_n$, B has rows $v_1, \ldots, v_i, \ldots, v_n$, and C has rows $v_1, \ldots, v_i', \ldots, v_n$,
Given an n-permutation σ, we have:

$$A_{k\sigma_k} = \begin{cases} B_{k\sigma_k} = C_{k\sigma_k} & \text{when } k \neq i, \\ B_{i\sigma_i} + C_{i\sigma_i} & \text{when } k = i. \end{cases}$$

Therefore

$$\begin{aligned} &\operatorname{sign}(\sigma)A_{1\sigma_1}\cdots A_{n\sigma_n} \\ &\qquad = \operatorname{sign}(\sigma)A_{1\sigma_1}\cdots A_{i-1\sigma_{i-1}}(B_{i\sigma_i} + C_{i\sigma_i})A_{i+1\sigma_{i+1}}\cdots A_{n\sigma_n} \\ &\qquad = \operatorname{sign}(\sigma)B_{1\sigma_1}\cdots B_{n\sigma_n} + \operatorname{sign}(\sigma)C_{1\sigma_1}\cdots C_{n\sigma_n}. \end{aligned}$$