Slides for April 25

MATH 24 — SPRING 2014

Permutation Matrices

Definition

A **permutation matrix** is a square matrix with exactly one 1 in each row and in each column, and all other entries are 0.

$$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} \quad \begin{pmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{pmatrix} \quad \begin{pmatrix}0 & 0 & 1\\1 & 0 & 0\\0 & 1 & 0\end{pmatrix}$$

$$\begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{pmatrix} \quad \begin{pmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{pmatrix}$$

If P is a permutation matrix and A any matrix with matching size, then PA has the same rows as A but in a different order.

$$\boxed{\text{The inverse of } P \text{ is the transpose } P^t}$$

LU-Decomposition

Theorem

For every invertible square matrix A, there are

- *a permutation matrix P,*
- *an upper triangular matrix U with no zeros along the diagonal, and*
- *a lower triangular matrix L with all 1's along the diagonal,*

such that $A = PLU$.

$$\begin{pmatrix} -1 & -1 & -5 \\ 1 & 7 & -2 \\ 1 & -5 & 15 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 & -5 \\ 0 & 6 & -7 \\ 0 & 0 & 3 \end{pmatrix}$$

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matlab: [L U P] = lu(A)
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LU-Usage

$$A = \begin{pmatrix} -1 & -1 & -5 \\ 1 & 7 & -2 \\ 1 & -5 & 15 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 & -5 \\ 0 & 6 & -7 \\ 0 & 0 & 3 \end{pmatrix} = LU$$

To solve $Ax = b = (-4, 9, 2)$:

1. Solve $Ly = (-4, 9, 2)$ by forward substitution:

$$y_1 = -4, \quad y_2 = 9 + y_1 = 5, \quad y_3 = 2 + y_1 + y_2 = 3.$$

2. Solve $Ux = y = (-4, 5, 3)$ by backward substitiution:

$$\begin{aligned} x_3 &= 3/3 = 1, \\ x_2 &= (5 + 7x_3)/6 = 2, \\ x_1 &= (-4 + x_2 + 5x_3)/(-1) = -3. \end{aligned}$$

```
matlab: x = U \ (L \ b)
```

LU-Usage

The main purpose of the LU-decomposition is to avoid computing inverses! But if you must compute an inverse, the LU-decomposition may be helpful...

The matrix A and its inverse A^{-1} are related by

$$Ax = y \quad \text{if and only if} \quad A^{-1}y = x.$$

So the columns of the inverse

$$x_1 = A^{-1}e_1, \quad x_2 = A^{-1}e_2, \quad \dots, \quad x_n = A^{-1}e_n$$

can be computed by solving

$$Ax_1 = e_1, \quad Ax_2 = e_2, \quad \dots, \quad Ax_n = e_n.$$

These are easier to solve given a LU-decomposition $A = LU$...

LU-Usage

$$A = \begin{pmatrix} -1 & -1 & -5 \\ 1 & 7 & -2 \\ 1 & -5 & 15 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 & -5 \\ 0 & 6 & -7 \\ 0 & 0 & 3 \end{pmatrix} = LU$$

- $L(0,0,1) = (0,0,1)$ and $U(-37/18, 7/18, 1/3) = (0,0,1)$.
- $L(0,1,1) = (0,1,0)$ and $U(-20/9, 5/9, 1/3) = (0,1,1)$.
- $L(1,1,2) = (1,0,0)$ and $U(-95/18, 17/18, 2/3) = (0,1,1)$.

$$A^{-1} = \frac{1}{18} \begin{pmatrix} -95 & -40 & -37 \\ 17 & 10 & 7 \\ 12 & 6 & 6 \end{pmatrix}$$

LU-Computation

- Reduce $A \rightsquigarrow U$ to echelon form using only lower triangular type 3 elementary matrices E on the left.
- Update $I \rightsquigarrow L$ by multiplying on the right by E^{-1} at each step.

$$
\left(\begin{array}{c|c} A & I \end{array} \right) = \left(\begin{array}{ccc|ccc} \color{red}{-1} & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -2 & -4 & 0 & 0 & 1 \end{array} \right)
$$

$$
\left(\begin{array}{ccc|ccc} -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & \color{red}{1} & 0 & 0 & 1 & 0 \\ 0 & -2 & -5 & -1 & 0 & 1 \end{array} \right)
$$

$$
\left(\begin{array}{ccc|ccc} -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \color{red}{-5} & -1 & -2 & 1 \end{array} \right) = \left(\begin{array}{c|c} U & L \end{array} \right)
$$

LU-Computation

If you get stuck, reorder the rows using a permutation matrix...

$$\left(\begin{array}{c|c} A & I \end{array}\right) = \left(\begin{array}{ccc|ccc} \color{red}{1} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \color{red}{0} & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right) \quad \textit{Oops}!$$

Swap the last two rows using $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

LU-Computation

If you get stuck, reorder the rows using a permutation matrix...

$$
\left(\; PA \mid I \; \right) = \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right)
$$

$$
\left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right)
$$

$$
\left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right) = \left(\; U \mid L \; \right)
$$

Then $PA = LU$ or, equivalently, $A = P^t LU$.