Slides for April 16

Math 24 — Spring 2014

Inverse Transformations

Suppose $T : \mathsf{V} \to \mathsf{W}$ and $S : \mathsf{W} \to \mathsf{V}$ are linear transformations.

- S is a **left inverse** of T if $ST = I_{\mathsf{V}}$.
- S is a **right inverse** of T if $TS = I_{\mathsf{W}}$.

Theorem

If the linear transformation $T : \mathsf{V} \to \mathsf{W}$ has both a left inverse and a right inverse, then they are the same.

Proof.

If $ST = I_{\mathsf{V}}$ and $TS' = I_{\mathsf{W}}$ then

$$S = SI_{\mathsf{W}} = S(TS') = (ST)S' = I_{\mathsf{V}}S' = S'.$$

□

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Theorem

If the linear transformation $T : \mathsf{V} \to \mathsf{W}$ has both a left inverse and a right inverse, then they are the same.

Definition

When it exists, the **inverse** of a linear transformation $T : \mathsf{V} \to \mathsf{W}$ is the unique linear transformation $T^{-1} : \mathsf{W} \to \mathsf{V}$ such that both $T^{-1}T = I_{\mathsf{V}}$ and $TT^{-1} = I_{\mathsf{W}}$.

One-to-One, Onto, and Inverses

Theorem

If $T : \mathsf{V} \to \mathsf{W}$ and $S : \mathsf{W} \to \mathsf{V}$ are linear transformations such that

$$ST = I_{\mathsf{V}}$$

then

(a) *T is one-to-one, and*

(b) *S is onto.*

Proof.

(a) If $u, v \in \mathsf{V}$ are such that $T(u) = T(v)$ then $u = S(T(u)) = S(T(v)) = v$.

(b) Given any $v \in \mathsf{V}$, the vector $w = T(v)$ is such that $S(w) = S(T(v)) = v$. □

One-to-One, Onto, and Inverses

Theorem

If $T : \mathsf{V} \to \mathsf{W}$ *and* $S : \mathsf{W} \to \mathsf{V}$ *are linear transformations such that*

$$ST = I_{\mathsf{V}}$$

then

(a) T *is one-to-one, and*

(b) S *is onto.*

Corollary

Let $T : \mathsf{V} \to \mathsf{W}$ *be a linear transformation.*

(a) *If* T *has a left inverse then* T *is one-to-one.*

(b) *If* T *has a right inverse then* T *is onto.*

One-to-One Transformations

Theorem

Suppose V *and* W *are finite dimensional vector spaces. For any linear transformation* $T : \mathrm{V} \to \mathrm{W}$*, the following are equivalent:*

1. T *is one-to-one*
2. $\mathrm{nullity}(T) = 0$
3. $\mathrm{rank}(T) = \dim(\mathrm{V})$
4. T *has a left inverse*

We already know that 1, 2, 3 are equivalent.
We saw that 4 implies 1.
So it suffices to prove that 1 implies 4.

One-to-One Transformations

Proof of 1 $\to$ 4.

Suppose $T : \mathrm{V} \to \mathrm{W}$ is one-to-one. Let $\{x_1, \ldots, x_n\}$ be a basis for V. By Exercise 14(a) of §2.1, $\{y_1, \ldots, y_n\}$ is linearly independent where $y_1 = T(x_1), \ldots, y_n = T(x_n)$; extend this set to a basis $\{y_1, \ldots, y_m\}$ for W.

By Theorem 2.6, there is a linear transformation $S : \mathrm{W} \to \mathrm{V}$ such that $S(y_1) = x_1, \ldots, S(y_n) = x_n$ and $S(y_i) = 0$ for $n < i \leq m$.

If $x = a_1x_1 + \cdots + a_nx_n$ is any vector in V, then

$$\begin{aligned} S(T(v)) &= ST(a_1x_1 + \cdots + a_nx_n) \\ &= a_1ST(x_1) + \cdots + a_nST(x_n) \\ &= a_1S(y_1) + \cdots + a_nS(y_n) \\ &= a_1x_1 + \cdots + a_nx_n = x. \end{aligned}$$

Therefore $ST = I_{\mathrm{V}}$. □

Onto Transformations

Theorem

Suppose V *and* W *are finite dimensional vector spaces. For any linear transformation* $T : \mathrm{V} \to \mathrm{W}$, *the following are equivalent:*

1. *T is onto*
2. $\mathrm{rank}(T) = \dim(\mathrm{W})$
3. $\mathrm{nullity}(T) = \dim(\mathrm{V}) - \dim(\mathrm{W})$
4. *T has a right inverse*

We already know that 1, 2, 3 are equivalent.
We saw that 4 implies 1.
So it suffices to prove that 1 implies 4.

Onto Transformations

Proof of $1 \to 4$.

Suppose $T : \mathrm{V} \to \mathrm{W}$ is onto. Let $\{y_1, \ldots, y_n\}$ be a basis for W. Since T is onto, we can pick matching $\{x_1, \ldots, x_n\}$ in V such that

$$T(x_1) = y_1, \ldots, T(x_n) = y_n.$$

By Theorem 2.6, there is a linear transformation $S : \mathrm{W} \to \mathrm{V}$ such that $S(y_1) = x_1, \ldots, S(y_n) = x_n$.
If $y = a_1 y_1 + \cdots + a_n y_n$ is any vector in W, then

$$\begin{aligned} T(S(y)) &= TS(a_1 y_1 + \cdots + a_n y_n) \\ &= a_1 TS(y_1) + \cdots + a_n TS(y_n) \\ &= a_1 T(x_1) + \cdots + a_n T(x_n) \\ &= a_1 y_1 + \cdots + a_n y_n = y. \end{aligned}$$

Therefore $TS = I_{\mathrm{W}}$. □

Isomorphisms

Theorem

Suppose V *and* W *are finite dimensional vector spaces of equal dimension* n *with ordered bases* α *and* β*, respectively. For any linear transformation* $T : \mathrm{V} \to \mathrm{W}$*, the following are equivalent:*

- T *is one-to-one and onto*
- T *is one-to-one*
- T *is onto*
- $\text{nullity}(T) = 0$
- *The columns of* $[T]_\alpha^\beta$ *are independent*
- T *is invertible*
- T *has a left inverse*
- T *has a right inverse*
- $\text{rank}(T) = n$
- *The columns of* $[T]_\alpha^\beta$ *generate* F^n