Slides for April 14

Math 24 — Spring 2014

The Space $\mathcal{L}(\mathsf{V}, \mathsf{W})$

The collection $\mathcal{L}(\mathsf{V}, \mathsf{W})$ of all linear transformations from V to W is a vector space where:

- The sum $S + T : \mathsf{V} \to \mathsf{W}$ is defined by
$$(S + T)(x) = S(x) + T(x)$$
for all $x \in \mathsf{V}$.
- The scalar product $aT : \mathsf{V} \to \mathsf{W}$ is defined by
$$(aT)(x) = aT(x)$$
for all $x \in \mathsf{V}$.
- The zero transformation $T_0 : \mathsf{V} \to \mathsf{W}$ is defined by
$$T_0(x) = 0$$
for all $x \in \mathsf{V}$.

Algebra of Linear Transformations

If $S \in \mathcal{L}(\mathrm{V}, \mathrm{W}),\ T, T' \in \mathcal{L}(\mathrm{W}, \mathrm{X}),\ U \in \mathcal{L}(\mathrm{X}, \mathrm{Y})$ then:

Identity

$$SI_{\mathrm{V}} = S \quad \text{and} \quad I_{\mathrm{W}}S = S$$

Associativity

$$(UT)S = U(TS)$$

Distributivity

$$(T + T')S = TS + T'S$$

$$U(T + T') = UT + UT'$$

Scalars Commute

$$(cT)S = c(TS) = T(cS)$$

From $\mathrm{M}_{m\times n}(F)$ to $\mathcal{L}(F^n, F^m)$

Let α and β be the standard ordered bases for F^n and F^m, respectively.

If A is a $m \times n$ matrix then the function $L_A : F^n \to F^m$ defined by

$$L_A(x) = Ax$$

for $x \in F^n$ is a linear transformation such that $A = [L_A]_\alpha^\beta$.

The correspondence

$$A \in \mathrm{M}_{m\times n}(F) \mapsto L_A \in \mathcal{L}(F^n, F^m)$$

is a linear transformation:

$$L_{A+B} = L_A + L_B, \quad L_{cA} = cL_A, \quad L_O = T_0.$$

From $\mathcal{L}(F^n, F^m)$ to $\mathsf{M}_{m\times n}(F)$

Let α and β be the standard ordered bases for F^n and F^m, respectively.

If $T : F^n \to F^m$ is a linear transformation then $[T]_\alpha^\beta$ is a $m \times n$ matrix such that

$$[T(x)]_\beta = [T]_\alpha^\beta x$$

for every $x \in F^n$.

The correspondence

$$T \in \mathcal{L}(F^n, F^m) \mapsto [T]_\alpha^\beta \in \mathsf{M}_{m\times n}(F)$$

is a linear transformation:

$$[S + T]_\alpha^\beta = [S]_\alpha^\beta + [T]_\alpha^\beta, \quad [aT]_\alpha^\beta = a[T]_\alpha^\beta, \quad [T_0]_\alpha^\beta = O.$$

Algebra of Matrices

If $A \in \mathrm{M}_{p\times q}(F),\ B, B' \in \mathrm{M}_{q\times r}(F),\ C \in \mathrm{M}_{r\times s}(F)$ then:

Identity

$$AI_q = A \quad \text{and} \quad I_pA = A$$

Associativity

$$(AB)C = A(BC)$$

Distributivity

$$A(B + B') = AB + AB'$$

$$(B + B')C = BC + B'C$$

Scalars Commute

$$(cA)B = c(AB) = A(cB)$$