Quiz 4

MATH 24 — SPRING 2014

Sample Solutions

Given

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -4 & -3 & -4 \\ 0 & -5 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

Compute A^{-1}.

Solution. The colums of A^{-1} are $v_1 = A^{-1}e_1$, $v_2 = A^{-1}e_2$ and $v_3 = A^{-1}e_1$. In other words, they are the solutions to the equations $Av_1 = e_1$, $Av_2 = e_2$ and $Av_3 = e_3$. Since we are given the LU-factorization of A, we can solve these with ease…

- To solve $Av_1 = e_1$, first solve $Lw_1 = e_1$ to find $w_1 = (1, 4, 20)$ and then solve $Uv_1 = w_1$ to find $v_1 = (11/3, 4, -20/3)$.
- To solve $Av_2 = e_2$, first solve $Lw_2 = e_2$ to find $w_2 = (0, 1, 5)$ and then solve $Uv_2 = w_2$ to find $v_2 = (2/3, 1, -5/3)$.
- To solve $Av_3 = e_3$, first solve $Lw_3 = e_3$ to find $w_3 = (0, 0, 1)$ and then solve $Uv_3 = w_3$ to find $v_3 = (1/3, 0, -1/3)$.

Therefore,

$$A^{-1} = \begin{pmatrix} 11/3 & 2/3 & 1/3 \\ 4 & 1 & 0 \\ -20/3 & -5/3 & -1/3 \end{pmatrix}. \qquad \square$$

Solution. An alternative is to find the reduced row echelon form of the augmented matrix

$$\left(\; A \mid I \; \right) = \left(\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ -4 & -3 & -4 & 0 & 1 & 0 \\ 0 & -5 & -3 & 0 & 0 & 1 \end{array} \right).$$

This can be done in using five row operations to obtain

$$\left(\; I \mid A^{-1} \; \right) = \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 11/3 & 2/3 & 1/3 \\ 0 & 1 & 0 & 4 & 1 & 0 \\ 0 & 0 & 1 & -20/3 & -5/3 & -1/3 \end{array} \right). \qquad \square$$

Solution. Yet another alternative is to compute L^{-1} and U^{-1} since $A^{-1} = U^{-1}L^{-1}$ from Exercise 4 of Section 2.4. Finding the inverse of an upper/lower triangular matrix is significantly easier than the general case.
We already did all the work to find L^{-1} in the first solution by computing the three vectors w_1, w_2, w_3, so

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 20 & 5 & 1 \end{pmatrix}.$$

Finding U^{-1} involves finding x_1, x_2, x_3 such that $Ux_1 = e_1$, $Ux_2 = e_2$, $Ux_3 = e_3$. Using a similar process to the first solution, we find $x_1 = (1, 0, 0)$, $x_2 = (-1, 1, 0)$, $x_3 = (1/3, 0, -1/3)$. So

$$U^{-1} = \begin{pmatrix} 1 & -1 & 1/3 \\ 0 & 1 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}.$$

Finally,

$$A^{-1} = U^{-1}L^{-1} = \begin{pmatrix} 11/3 & 2/3 & 1/3 \\ 4 & 1 & 0 \\ -20/3 & -5/3 & -1/3 \end{pmatrix}.$$

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