Quiz 3

MATH 24 — SPRING 2014

Sample Solutions

The space $\mathsf{P}_2(\mathbb{R})$ has for standard ordered basis $\alpha = \{1, x, x^2\}$ and another ordered basis

$$\beta = \left\{ \tfrac{1}{2}x^2 - \tfrac{1}{2}x, 1 - x^2, \tfrac{1}{2}x^2 + \tfrac{1}{2}x \right\}.$$

The space $\mathbb{R}^3$ has the standard ordered basis $\gamma = \{e_1, e_2, e_3\}$.
Let $T : \mathsf{P}_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformation defined by $T(f) = (f(-1), f(0), f(1))$ for every $f(x) \in \mathsf{P}_2(\mathbb{R})$.
For example,

$$T(x^2 + x + 1) = \begin{pmatrix} (-1)^2 + (-1) + 1 \\ (0)^2 + (0) + 1 \\ (1)^2 + (1) + 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$

(a) Compute $[x^2]_\beta, [x]_\beta, [1]_\beta$. Justify your answers.

Solution. Since

$$x^2 = 1(\tfrac{1}{2}x^2 - \tfrac{1}{2}x) + 0(1 - x^2) + 1(\tfrac{1}{2}x^2 + \tfrac{1}{2}x),$$

we see that $[x^2]_\beta = (1, 0, 1)$.

Since

$$x = -1(\tfrac{1}{2}x^2 - \tfrac{1}{2}x) + 0(1 - x^2) + 1(\tfrac{1}{2}x^2 + \tfrac{1}{2}x),$$

we see that $[x]_\beta = (-1, 0, 1)$.

Since

$$1 = 1(\tfrac{1}{2}x^2 - \tfrac{1}{2}x) + 1(1 - x^2) + 1(\tfrac{1}{2}x^2 + \tfrac{1}{2}x),$$

we see that $[1]_\beta = (1, 1, 1)$. □

(b) Compute $[T]_\alpha^\gamma$ and $[T]_\beta^\gamma$. Justify your answers.

Solution. Since $T(1) = (1, 1, 1), T(x) = (-1, 0, 1), T(x^2) = (1, 0, 1)$, we see that

$$[T]_\alpha^\gamma = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Since $T(\frac{1}{2}x^2 - \frac{1}{2}x) = (1, 0, 0), T(1 - x^2) = (0, 1, 0), T(\frac{1}{2}x^2 + \frac{1}{2}x) = (0, 0, 1)$, we see that

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ □

(c) Show that $[T]_\alpha^\gamma [f]_\alpha = [f]_\beta$ for every $f(x) \in \mathsf{P}_2(\mathbb{R})$.

Solution. By Theorem 2.14, we have

$$[T(f)]_\gamma = [T]_\alpha^\gamma [f]_\alpha \quad \text{and} \quad [T(f)]_\gamma = [T]_\beta^\gamma [f]_\beta.$$

Therefore,

$$[T]_\alpha^\gamma [f]_\alpha = [T]_\beta^\gamma [f]_\beta.$$

Since $[T]_\beta^\gamma$ is the 3×3 identity matrix by part (b), we also have $[T]_\beta^\gamma [f]_\beta = [f]_\beta$. Therefore, $[T]_\alpha^\gamma [f]_\alpha = [f]_\beta$. □

Solution. Suppose $f(x) = a + bx + cx^2$. By part (a), we see that

$$[f]_\beta = [a1 + bx + cx^2]_\beta = a[1]_\beta + b[x]_\beta + c[x^2]_\beta = \begin{pmatrix} a - b + c \\ a \\ a + b + c \end{pmatrix}.$$

On the other hand,

$$[T]_\alpha^\gamma [f]_\alpha = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a - b + c \\ a \\ a + b + c \end{pmatrix}.$$

Therefore, $[T]_\alpha^\gamma [f]_\alpha = [f]_\beta$. □