Quiz 3

MATH 24 — SPRING 2014

Sample Solutions

The space $\mathsf{P}_2(\mathbb{R})$ has the standard ordered basis $\alpha = \{1, x, x^2\}$ and the space $\mathbb{R}^2$ has the standard ordered basis $\gamma = \{e_1, e_2\}$.
Let $T : \mathsf{P}_2(\mathbb{R}) \to \mathbb{R}^2$ be the linear transformation defined by $T(f(x)) = (f'(0), f'(1))$ for every $f(x) \in \mathsf{P}_2(\mathbb{R})$. For example,

$$T(x^2 - x) = \begin{pmatrix} 2(0) - 1 \\ 2(1) - 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

since the derivative of $x^2 - x$ is $2x - 1$.
Let $S : \mathbb{R}^2 \to \mathsf{P}_2(\mathbb{R})$ be the linear transformation such that

$$[S]_\gamma^\alpha = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 0 \end{pmatrix}.$$

(a) Compute $S(e_1)$ and $S(e_2)$. Justify your answers.

Solution. Since

$$[S(e_1)]_\alpha = [S]_\gamma^\alpha [e_1]_\gamma = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

we see that

$$S(e_1) = 1 - x.$$

Since

$$[S(e_2)]_\alpha = [S]_\gamma^\alpha [e_2]_\gamma = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

we see that

$$S(e_2) = x.$$

□

(b) Compute $[T]_\alpha^\gamma$. Justify your answer.

Solution. Since $T(1) = (0, 0), T(x) = (1, 1), T(x^2) = (0, 2)$,

$$[T]_\alpha^\gamma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

□

(c) Show that $ST(f(x)) = f'(x)$ for every $f(x) \in \mathsf{P}_2(\mathbb{R})$.

Solution. Since

$$\begin{aligned} ST(1) &= S(0,0) = 0 \\ ST(x) &= S(1,1) = S(e_1) + S(e_2) = (1-x) + (x) = 1 \\ ST(x^2) &= S(0,2) = 2S(e_2) = 2x, \end{aligned}$$

We see that if $f(x) = a_0 + a_1 x + a_2 x^2$ then

$$\begin{aligned} ST(f) &= a_0 ST(1) + a_1 ST(x) + a_2 ST(x^2) \\ &= a_0(0) + a_1(1) + a_2(2x) = a_1 + 2a_2 x = f'(x). \end{aligned}$$

□

Solution. First note that, by Theorem 2.11, we have

$$[ST]_\alpha^\alpha = [S]_\gamma^\alpha [T]_\alpha^\gamma = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\frac{d}{dx}[1] = 0$, $\frac{d}{dx}[x] = 1$, $\frac{d}{dx}[x^2] = 2x$, this is also the matrix representation of the derivative with respect to the standard basis α. By Theorem 2.20, we conclude that $ST(f) = f'$ for every $f \in \mathsf{P}_2(\mathbb{R})$. □