Quiz 2

MATH 24 — SPRING 2014

Sample Solutions

Find a basis for the subspace W of $\mathbb{R}^3$ consisting of all vectors (x_1, x_2, x_3) such that $x_1 + 3x_2 - 2x_3 = 0$.
Justify your answer.

Solution. The given equation can be rewritten $x_1 = 2x_3 - 3x_2$. Given $x_2 = a$ and $x_3 = b$, we are led to the vector solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2b - 3a \\ a \\ b \end{pmatrix} = a \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

So the two vectors $(-3, 1, 0)$ and $(2, 0, 1)$ generate W.
These two vectors are linearly independent since

$$a \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

implies that $a = 0$ by looking at the second coordinate and that $b = 0$ by looking at the third coordinate.
Therefore $\{(-3, 1, 0), (2, 0, 1)\}$ is a basis for the subspace W. □

Solution. By inspection, we find the two solutions $(1, 1, 2)$ and $(5, -1, 1)$. These are linearly independent vectors because

$$a \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + b \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

leads to the system of equations

$$a + 5b = 0, \quad a - b = 0, \quad 2a + b = 0.$$

The second equation entails that $a = b$ and then the third equation leads to $3a = 0$. Since we're working in the field $\mathbb{R}$, we know that $3 \neq 0$ and therefore $a = b = 0$.
Since we found two linearly independent vectors in W, we know that the dimension of W is at least 2. The dimension of W cannot be 3 for then W would have to equal $\mathbb{R}^3$ but we know that $W \neq \mathbb{R}^3$ since, for example, the vector $(1, 1, 1)$ is not in W.
Therefore, W must have dimension 2 and this means that $\{(5, -1, 1), (1, 1, 2)\}$ is a basis for W by Corollary 2(b) of Theorem 1.10. □