Quiz 1

MATH 24 — SPRING 2014

Sample Solutions

A $n \times n$ matrix A over the field $\mathbb{R}$ of real numbers is **skew-symmetric** if $A^t = -A$. (The transpose A^t of a matrix A is defined on page 17.) By Exercise 28 of Section 1.3, we know that the set W_n of all $n \times n$ skew-symmetric matrices over $\mathbb{R}$ forms a subspace of $\mathsf{M}_{n\times n}(\mathbb{R})$.

Show that the space W_2 of all 2×2 skew-symmetric matrices over $\mathbb{R}$ is generated by the single matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Solution. We need to show that $\mathsf{W}_2 = \operatorname{span}\{B\}$, where

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This amounts to two inclusions $\mathsf{W}_2 \subseteq \operatorname{span}\{A\}$ and $\operatorname{span}\{A\} \subseteq \mathsf{W}_2$.

To see that $\operatorname{span}\{B\} \subseteq \mathsf{W}_2$, it suffices to observe that B is skew-symmetric. This is because $\operatorname{span}\{B\}$ is the smallest subspace of $\mathsf{M}_{2\times 2}(\mathbb{R})$ that contains B, so if $B \in \mathsf{W}_2$ then it must be that $\operatorname{span}\{B\} \subseteq \mathsf{W}_2$.

To see that $\mathsf{W}_2 \subseteq \operatorname{span}\{B\}$, we need to show that every 2×2 skew-symmetric matrix is a scalar multiple of B. In order for the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to be skew-symmetric, we need $a = -a$, $d = -d$ and $b = -c$. Since we are working over the field $\mathbb{R}$, we can conclude that $a = d = 0$. Therefore,

$$A = \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix} = cB,$$

which shows that A is a scalar multiple of B. Since A was an arbitrary skew-symmetric 2×2 matrix, we conclude that $\mathsf{W}_2 \subseteq \operatorname{span}\{B\}$. ☐