Homework Notes — Week 6

Math 24 — Spring 2014

§3.4#4b The system

$$\begin{aligned} x_1 + x_2 - 3x_3 + x_4 &= -2, \\ x_1 + x_2 + x_3 - x_4 &= 2, \\ x_1 + x_2 - x_3 &= 0, \end{aligned}$$

is consistent. To see this we put the matrix

$$(A|b) = \begin{pmatrix} 1 & 1 & -3 & 1 & -2 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 0 & 0 \end{pmatrix},$$

into reduced row echelon form.

Adding -1 times the first row to the second and third rows we obtain the matrix

$$\begin{pmatrix} 1 & 1 & -3 & 1 & -2 \\ 0 & 0 & 4 & -2 & 4 \\ 0 & 0 & 2 & -1 & 2 \end{pmatrix}.$$

Now we note that the second row is twice the third, so we can eliminate the third row and then multiply the second row by $1/4$ to obtain the matrix

$$\begin{pmatrix} 1 & 1 & -3 & 1 & -2 \\ 0 & 0 & 1 & -1/2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now adding three times the second row to the first we obtain the matrix

$$\begin{pmatrix} 1 & 1 & 0 & -1/2 & 1 \\ 0 & 0 & 1 & -1/2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now since this is in reduced row echelon form exercise 3 says that the system is consistent, i.e. that it has a solution. To see what's it's solution set is, we just read off what an arbitrary solution must be from the above matrix, namely that in order for (x_1, x_2, x_3, x_4) to be a solution we must have

$$\begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{pmatrix} -x_2 + \frac{1}{2}x_4 + 1 \\ x_2 \\ \frac{1}{2}x_4 + 1 \\ x_4 \end{pmatrix}, \\
&= x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.
\end{aligned}$$

Now finally by Theorem 3.15(b) the set

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 1 \end{pmatrix} \right\}$$

is a basis for the corresponding homogeneous system.

§3.4#15* **Theorem.** *A matrix has only one reduced row echelon form.*

Proof. We will prove this by induction on n, the number of columns of a matrix. For this, let A be an $m \times n$ matrix and suppose that B, C are $m \times n$ matrices which are both reduced row echelon forms of A.

If $n = 1$ then A, B, C are all just column vectors and we really only have two options, $\operatorname{rank} A = 0$ or 1. If $\operatorname{rank} A = 0$ then $A = B = C = 0$ so in particular $B = C$. If $\operatorname{rank} A = 1$ then by Theorem 3.16 we know that there is a column of B that is the column vector e_1, i.e. that $B = e_1$. But similar $C = e_1$, hence $B = C$.

Now suppose that $n > 1$ and that all matrices with $n - 1$ columns have a unique reduced row echelon form. Let A', B', C' be the matrices obtained by deleting the nth column of A, B, C respectively. Since deleting a column has no effect on the row operations in the remaining columns, B', C' are reduced row echelon forms of A' and so by the induction hypothesis we have that $B' = C'$.

So at the very least the first $n - 1$ columns of B and C are the same, so the only way they could disagree is in column n. Note that adding a column to a matrix either leaves the rank the same or increases it by one we have that $\operatorname{rank} A = \operatorname{rank} A'$ or $\operatorname{rank} A = \operatorname{rank} A' + 1$. Throughout the rest let $r = \operatorname{rank} A$.

If $r = \operatorname{rank} A'$ then by Theorem 3.16 part (b) there are columns $b_{j_i} = e_i$ of B for each $i = 1, 2, \ldots, r$. Now since $B' = C'$ we know that the columns $c_{j_i} = e_i$ as well. Since column n of B is of the form $d_1 e_1 + \cdots d_r e_r$ for some scalars $d_1, \ldots d_r$ we know that column n of A is $d_1 a_{j_1} + \cdots d_r a_{j_r}$, this is from Theorem 3.16 part (d). But similarly column n of C is of the form $d'_1 e_1 + \cdots d'_r e_r$ for some scalars $d'_1, \ldots, d'_r$ and again column n of A is $d'_1 a_{j_1} + \cdots d'_r a_{j_r}$.

But Theorem 3.16 part (c) says that $\{a_{j_1}, \ldots, a_{j_r}\}$ is linearly independent, and we have just shown that

$$d_1 a_{j_1} + \cdots d_r a_{j_r} = d'_1 e_1 + \cdots d'_r e_r,$$

so by linear independence of these vectors we have $d_1 = d'_1, d_2 = d'_2, \ldots, d_r = d'_r$. But this means exactly that the nth column of B and C are the same, so finally $B = C$

If $r = \operatorname{rank} A' + 1$ then we will see that the column n of B and C is precisely e_r. Since B', C' are in reduced row echelon form and have rank $r - 1$ they must have all zeros in row r. But since B, C have rank r there must be a nonzero entry in row r of these matrices, in particular the entry of column n and row r of B and C must be nonzero. Since this is the first nonzero entry in row r it must be 1 and this must be the only nonzero entry in column n as B and C are reduced row echelon forms. But this means precisely that the nth columns of B and C are e_r, hence that $B = C$. $\square$

§4.1#9 For any $A, B \in M_{2\times 2}(F)$ we have $\det(AB) = \det(A)\det(B)$. We know that we can write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} e & f \\ g & h \end{pmatrix},$$

for some scalars $a, b, \ldots, h$. By definition

$$\det(A)\det(B) = (ad - bc)(eh - fg),$$

and we can compute $\det(AB)$ directly,

$$\begin{aligned}
\det(AB) &= \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right), \\
&= \det\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} \\
&= (ae+bg)(cf+dh)-(af+bh)(ce+dg), \\
&= aecf+aedh+bgcf+bgdh-afce-afdg-bhce-bhdg, \\
&= acef-acef+adeh-afdg+bcgf-bche+bdgh-bdgh, \\
&= adeh-afdg+bcgf-bche, \\
&= ad(eh-fg)-bc(eh-fg), \\
&= (ad-bc)(eh-fg).
\end{aligned}$$

§4.1#10 Let $A \in M_{2\times 2}(F)$ and C the classical adjoint of A, i.e.

$$C = \begin{pmatrix} A_{2,2} & -A_{1,2} \\ -A_{2,1} & A_{1,1} \end{pmatrix}.$$

For part (a) we compute directly AC and CA,

$$\begin{aligned}
AC &= \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}\begin{pmatrix} A_{2,2} & -A_{1,2} \\ -A_{2,1} & A_{1,1} \end{pmatrix}, \\
&= \begin{pmatrix} A_{1,1}A_{2,2}-A_{1,2}A_{2,1} & -A_{1,1}A_{1,2}+A_{1,2}A_{1,1} \\ A_{2,1}A_{2,2}-A_{2,2}A_{2,1} & -A_{2,1}A_{1,2}+A_{2,2}A_{1,1} \end{pmatrix}, \\
&= \begin{pmatrix} \det(A) & 0 \\ 0 & \det(A) \end{pmatrix}, \\
&= \det(A)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

The computation for CA is essentially identical.

For part (b) we just compute $\det(C)$,

$$\begin{aligned}
\det(C) &= \det\begin{pmatrix} A_{2,2} & -A_{1,2} \\ -A_{2,1} & A_{1,1} \end{pmatrix}, \\
&= A_{2,2}A_{1,1}-A_{1,2}A_{2,1}, \\
&= A_{1,1}A_{2,2}-A_{1,2}A_{2,1}, \\
&= \det(A).
\end{aligned}$$

Finally for part (c) we know that if $\det(A) \neq 0$ if and only if A is invertible, so if A is invertible then $A\left(\frac{1}{\det(A)}C\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and by uniqueness of inverses we have $A^{-1} = [\det(A)]^{-1}C$.

§4.1#11* **Theorem.** *Suppose $\delta : \mathsf{M}_{2\times 2}(F) \to F$ is a function with the following three properties.*

(i) *δ is a linear function of each row of the matrix when the other row is held fixed.*

(ii) *If the two rows of $A \in \mathsf{M}_{2\times 2}(F)$ are identical, then $\delta(A) = 0$.*

(iii) *If I is the 2×2 identity matrix, then $\delta(I) = 1$.*

Then $\delta(A) = \det(A)$ for every $A \in \mathsf{M}_{2\times 2}(F)$.

Suppose δ satisfies properties (i), (ii) and (iii). For the current proof, given vectors $x, y \in F^2$, let's write $\begin{pmatrix} x \\ y \end{pmatrix}$ for the 2×2 matrix whose rows are x and y as there will never be any danger to confuse this notation with that of a column vector. We will break down the argument in three lemmas.

Lemma 1. $\delta\begin{pmatrix} x \\ y \end{pmatrix} = -\delta\begin{pmatrix} y \\ x \end{pmatrix}$.

Proof. Because of property (i), we have

$$\delta\begin{pmatrix} x+y \\ x+y \end{pmatrix} = \delta\begin{pmatrix} x \\ x+y \end{pmatrix} + \delta\begin{pmatrix} y \\ x+y \end{pmatrix} = \delta\begin{pmatrix} x \\ x \end{pmatrix} + \delta\begin{pmatrix} x \\ y \end{pmatrix} + \delta\begin{pmatrix} y \\ x \end{pmatrix} + \delta\begin{pmatrix} y \\ y \end{pmatrix}.$$

Because of property (ii), we have

$$\delta\begin{pmatrix} x \\ x \end{pmatrix} = 0, \quad \delta\begin{pmatrix} y \\ y \end{pmatrix} = 0, \quad \text{and} \quad \delta\begin{pmatrix} x+y \\ x+y \end{pmatrix} = 0.$$

Therefore, $\delta\begin{pmatrix} x \\ y \end{pmatrix} + \delta\begin{pmatrix} y \\ x \end{pmatrix} = 0$ or, equivalently, $\delta\begin{pmatrix} x \\ y \end{pmatrix} = -\delta\begin{pmatrix} y \\ x \end{pmatrix}$. ☐

Lemma 2. *We have* $\delta\begin{pmatrix}e_1\\e_1\end{pmatrix} = 0$ *and* $\delta\begin{pmatrix}e_1\\e_2\end{pmatrix} = 1$ *and therefore* $\delta\begin{pmatrix}e_1\\y\end{pmatrix} = y_2$ *for every* $y = (y_1, y_2) \in F^2$.

Proof. The first two facts are immediate consequences of properties (ii) and (iii), respectively. By property (i), we then also have

$$\delta\begin{pmatrix}e_1\\y\end{pmatrix} = \delta\begin{pmatrix}e_1\\y_1e_1 + y_2e_2\end{pmatrix} = y_1\delta\begin{pmatrix}e_1\\e_1\end{pmatrix} + y_2\delta\begin{pmatrix}e_1\\e_2\end{pmatrix} = y_1(0) + y_2(1) = y_2$$

for every vector $y = (y_1, y_2) \in F^2$. □

Lemma 3. *We have* $\delta\begin{pmatrix}e_2\\e_1\end{pmatrix} = -1$ *and* $\delta\begin{pmatrix}e_2\\e_2\end{pmatrix} = 0$ *and therefore* $\delta\begin{pmatrix}e_2\\y\end{pmatrix} = -y_1$ *for every* $y = (y_1, y_2) \in F^2$.

Proof. The first fact follows from property (iii) and Lemma 1 and the second fact follows from property (ii). By property (i), we then also have

$$\delta\begin{pmatrix}e_2\\y\end{pmatrix} = \delta\begin{pmatrix}e_2\\y_1e_1 + y_2e_2\end{pmatrix} = y_1\delta\begin{pmatrix}e_2\\e_1\end{pmatrix} + y_2\delta\begin{pmatrix}e_1\\e_2\end{pmatrix} = y_1(-1) + y_2(0) = -y_1$$

for every vector $y = (y_1, y_2) \in F^2$. □

Now, by property (i), we have for any two vectors $x = (x_1, x_2), y = (y_1, y_2) \in F^2$ that

$$\begin{aligned}\delta\begin{pmatrix}x\\y\end{pmatrix} &= \delta\begin{pmatrix}x_1e_1 + x_2e_2\\y\end{pmatrix}\\ &= x_1\delta\begin{pmatrix}e_1\\y\end{pmatrix} + x_2\delta\begin{pmatrix}e_2\\y\end{pmatrix}\\ &= x_1y_2 - x_2y_1.\end{aligned}$$

It then follows immediately from the definition of 2×2 determinants that $\delta(A) = \det(A)$ for every 2×2 matrix A over F.

§4.2#23* **Theorem.** *The determinant of an upper triangular matrix is the product of its diagonal entries.*

Proof. This is easy to see from the alternate definition of determinants from the April 24 slides. The idea is that if A is an $n \times n$ matrix and σ is an n-permutation then

$$\operatorname{sign}(\sigma) A_{1\sigma_1} \cdots A_{n\sigma_n} = 0$$

unless σ is the permutation $(1, 2, \ldots, n)$, where every number is in order.

Indeed, given any other permutation σ, let i be the first number such that $\sigma_i \neq i$. We must then have $\sigma_j = i$ for some $j > i$. But then $A_{j\sigma_j} = A_{ji} = 0$ since A is upper triangular. □

Proof. Suppose that $A = (A_{i,j})_{1 \le i,j \le n}$ is an $n \times n$ upper triangular matrix, meaning that $A_{i,j} = 0$ for all $i > j$.

We prove this by induction on n, the number of columns (and rows) in an upper triangular matrix. If $n = 1$ then $A = (a)$ and $\det(A) = a$ by definition of det for 1×1 matrices. Now suppose that $n > 1$ and that the result holds for all $(n-1) \times (n-1)$ upper triangular matrices. Then we compute $\det(A)$ by cofactor expansion along row n. Theorem 4.4 then shows that

$$\det(A) = \sum_{j=1}^{n} (-1)^{n+j} A_{n,j} \cdot \det(\widetilde{A}_{n,n}) = (-1)^{n+n} A_{n,n} \det(\widetilde{A}_{n,n}) = A_{n,n} \det(\widetilde{A}_{n,n}),$$

since $A_{n,1} = A_{n,2} = \cdots = A_{n,n-1} = 0$. Now if $\widetilde{A}_{n,n}$ is an $(n-1) \times (n-1)$ matrix and is upper triangular with diagonal entries $A_{1,1}, A_{2,2}, \ldots, A_{n-1,n-1}$ so by the induction hypothesis

$$\det(A) = A_{n,n} \det(\widetilde{A}_{n,n}) = A_{n,n} A_{1,1} \cdots A_{n-1,n-1} = A_{1,1} \cdots A_{n,n},$$

which is precisely the product of the diagonal entries of A. □

§4.2#24 If an $n \times n$ matrix A has a row consisting entirely of zeros then $\det(A) = 0$.

First write $A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ for some row vectors $a_1, \ldots a_n$. Then suppose that $a_i = 0$ for some $1 \le i \le n$, then Theorem 4.3

$$
\begin{aligned}
\det(A) &= \det\begin{pmatrix} a_1 \\ \vdots \\ a_{i-1} \\ 0 \\ a_{i+1} \\ \vdots \\ a_n \end{pmatrix} \\
&= \det\begin{pmatrix} a_1 \\ \vdots \\ a_{i-1} \\ 0+0 \\ a_{i+1} \\ \vdots \\ a_n \end{pmatrix}, \\
&= \det\begin{pmatrix} a_1 \\ \vdots \\ a_{i-1} \\ 0 \\ a_{i+1} \\ \vdots \\ a_n \end{pmatrix} + \det\begin{pmatrix} a_1 \\ \vdots \\ a_{i-1} \\ 0 \\ a_{i+1} \\ \vdots \\ a_n \end{pmatrix}, \\
&= \det(A) + \det(A), \\
&= 2\det(A).
\end{aligned}
$$

So subtracting $\det(A)$ from both sides we get $\det(A) = 0$.

§4.3#11 If M is skew-symmetric, then $\det(-M) = \det(M^t) = \det(M)$ by Theorem 4.8. Since $-M = (-I)M$, we also have $\det(-M) = \det(-I)\det(M) = (-1)^n \det(M)$ by Theorem 4.7 and Exercise 23 from Section 4.2. If n is odd, then it follows from $\det(M) = -\det(M)$ that $\det(M) = 0$. (Since we are working over the complex numbers, $1 + 1 \neq 0$.)

If n is even, then the above says nothing much. A $(2k) \times (2k)$ a skew-symmetric matrix may or may not be invertible. The zero matrix is an example of a $(2k) \times (2k)$

skew-symmetric matrix that is not invertible. The matrix B given by

$$B = \begin{pmatrix} 0_k & I_k \\ -I_k & 0_k \end{pmatrix}$$

where I_k is the $k \times k$ identity matrix and 0_k is the $k \times k$ zero matrix, then B is invertible since $\operatorname{rank}(B) = 2k$ and B is clearly skew-symmetric.

Indeed, $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a skew-symmetric matrix with determinant 1.

§4.3#12 If $Q \in M_{n\times n}(\mathbb{R})$ is orthogonal then $\det(Q) = \pm 1$.

Recall a few facts about determinant, first we know that $\det(I) = 1$ (where I is the identity matrix), second $\det(AB) = \det(A)\det(B)$ for all $A, B \in M_{n\times n}(\mathbb{R})$ and third that $\det(A^t) = \det(A)$.

Using these facts we have

$$1 = \det(I) = \det(QQ^t) = \det(Q)\det(Q^t) = \det(Q)\det(Q) = \det(Q)^2,$$

but the only numbers in $\mathbb{R}$ whose squares are one are 1 and -1. Therefore $\det(Q) = \pm 1$.