Homework Notes — Week 4

Math 24 — Spring 2014

§2.4#4 Let A and B be $n \times n$ invertible matrices. We want to show that AB is invertible and that $(AB)^{-1} = B^{-1}A^{-1}$.

Recall that an $n \times n$ matrix X is *invertible* if there is an $n \times n$ matrix Y with $XY = YX = I$, the $n \times n$ identity matrix.

So to see that $X = AB$ is invertible we only need to find a matrix Y such that $(AB)Y = Y(AB) = I$. Well, the question actually tells us what matrix we should try to use for Y, namely $Y = B^{-1}A^{-1}$.

Now we actually need to see if this choice works, and to see this we recall a few facts, namely the fact that matrix multiplication is associative, and that

$$AA^{-1} = A^{-1}A = I \tag{1}$$

$$BB^{-1} = B^{-1}B = I. \tag{2}$$

Now we check that $(AB)Y = Y(AB) = I$,

$$\begin{aligned}
(AB)Y &= A(BY) && \text{(associativity of matrix multiplication)} \\
&= A(B(B^{-1}A^{-1})) && \text{(definition of } Y\text{)} \\
&= A((BB^{-1})A^{-1}) && \text{(associativity of matrix multiplication)} \\
&= A(IA^{-1}) && \text{(equation (2))} \\
&= AA^{-1} && \text{(} I \text{ is the identity matrix)} \\
&= I && \text{(equation (1))}
\end{aligned}$$

$$
\begin{aligned}
Y(AB) &= (YA)B && \text{(associativity of matrix multiplication)}\\
&= ((B^{-1}A^{-1})A)B && \text{(definition of } Y\text{)}\\
&= (B^{-1}(A^{-1}A))B && \text{(associativity of matrix multiplication)}\\
&= (B^{-1}I)B && \text{(equation (1))}\\
&= B^{-1}B && (I \text{ is the identity matrix})\\
&= I && \text{(equation (2)).}
\end{aligned}
$$

So we have seen that $(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})AB = I$, which means exactly that $(AB)^{-1} = B^{-1}A^{-1}$.

§2.4#7 Let A be an $n \times n$ matrix. We need to show that (a) if $A^2 = 0$ then A is not invertible, and (b) determine whether A could be invertible if $AB = 0$ for some nonzero $n \times n$ matrix B.

For (a), what would happen if A were invertible? In that case there exista some $n \times n$ matrix called A^{-1} such that $AA^{-1} = A^{-1}A = I$. An idea here would be to multiply both sides of $A^2 = 0$ by A^{-1} on the left to obtain a new equation, namely

$$
\begin{aligned}
0 &= A^{-1}0 && (0 = 0X = X0 \text{ for any matrix } X)\\
&= A^{-1}(A^2) && \text{(by our assumption)}\\
&= A^{-1}(AA) && (A^2 = AA)\\
&= (A^{-1}A)A && \text{(associativity of matrix multiplication)}\\
&= IA && (\text{since } A^{-1}A = I)\\
&= A && (I \text{ is the identity matrix}).
\end{aligned}
$$

But now $A = 0$ which is **NOT** and invertible matrix. So if A is invertible then $A^2 \neq 0$ which is the contrapositive of the statement we were asked to prove.

For (b), the answer is no. Suppose that A is invertible and that $AB = 0$ for some $n \times n$ matrix B, then we can demonstrate that in face $B = 0$, so there can be no nonzero B with the property that $AB = 0$.

Specifically we have that

$$
\begin{aligned}
B &= IB\\
&= (A^{-1}A)B\\
&= A^{-1}(AB)\\
&= A^{-1}0\\
&= 0,
\end{aligned}
$$

exactly as predicted.

§2.4#16* **Theorem.** *Let B be an invertible $n \times n$ matrix. The function $\Phi : \mathsf{M}_{n\times n}(F) \to \mathsf{M}_{n\times n}(F)$ defined by $\Phi(A) = B^{-1}AB$ is an isomorphism.*

Proof. First we need to verify that Φ is a linear transformation. Well if $C, D \in \mathsf{M}_{n\times n}(F)$ and $a \in F$ then

$$\begin{aligned}\Phi(aC + D) &= B^{-1}(aC + D)B \\ &= (B^{-1}(aC) + B^{-1}D)B \\ &= B^{-1}(aC)B + B^{-1}DB \\ &= aB^{-1}CB + B^{-1}DB \\ &= a\Phi(C) + \Phi(D),\end{aligned}$$

so Φ is a linear transformation.

Now we will find an inverse map to Φ. To this end, consider the map $\Psi : \mathsf{M}_{n\times n}(F) \to \mathsf{M}_{n\times n}(F)$ given by $\Psi(A) = BAB^{-1}$. A similar computation as with Φ shows that Ψ is a a linear transformation.

If $A \in \mathsf{M}_{n\times n}(F)$ then

$$\begin{aligned}\Phi \circ \Psi(A) &= \Phi(BAB^{-1}) \\ &= B^{-1}(BAB^{-1})B \\ &= (B^{-1}B)A(BB^{-1}) \\ &= I_nAI_n \\ &= A, \\ \Psi \circ \Phi(A) &= \Psi(B^{-1}AB) \\ &= B(B^{-1}AB)B^{-1} \\ &= (BB^{-1})A(BB^{-1}) \\ &= I_nAI_n \\ &= A,\end{aligned}$$

meaning exactly that $\Phi \circ \Psi = \Psi \circ \Phi = I_{\mathsf{M}_{n\times n}(F)}$, the identity transformation $\mathsf{M}_{n\times n}(F) \to \mathsf{M}_{n\times n}(F)$. □

§2.5#6bd For each matrix A and ordered basis β, we need to find find $[L_A]_\beta$ and also find an invertible matrix Q such that $[L_A]_\beta = Q^{-1}AQ$.

(b) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

For the first part of the problem we can just compute what $[L_A]_\beta$ using the definition. In particular

$$\begin{aligned}
L_A \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 3 \\ 3 \end{pmatrix} \\
&= 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
L_A \begin{pmatrix} 1 \\ -1 \end{pmatrix} &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
&= 0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ -1 \end{pmatrix},
\end{aligned}$$

so by definition $[L_A]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$.

To do this problem, we appeal to the Corollary to Theorem 2.23 on page 115 of the textbook which tells us that the matrix $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ satisfies the condition, i.e. that

$$[L_A]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} = Q^{-1}AQ.$$

(d) $A = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

Again we can compute $[L_A]_\beta$ just by computing L_A on the basis β,

$$
\begin{aligned}
L_A \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} &= \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \\
&= \begin{pmatrix} 6 \\ 6 \\ -12 \end{pmatrix} \\
&= 6 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \\
L_A \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} &= \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 12 \\ -12 \\ 0 \end{pmatrix} \\
&= 0 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + 12 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \\
L_A \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} &= \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 18 \\ 18 \\ 18 \end{pmatrix} \\
&= 0 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 18 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},
\end{aligned}
$$

so that by definition

$$
[L_A]_\beta = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{pmatrix}.
$$

Now again we can use the corollary on page 115 of the textbook to assert that if

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix}$$

then $[L_A]_\beta = Q^{-1}AQ$.

§2.5#9 We want to show that "is similar to" is an equivalence relation on $\mathsf{M}_{n\times n}(F)$.

By the definition of equivalence relations, this problem has three basic parts:

(i) Reflexivity: A is always similar to A.

(ii) Symmetry: If A is similar to B, then B is similar to A.

(iii) Transitivity: If A is similar to B and B is similar to C, then A is similar to C.

We proceed to verify all three in order.

(i) If $A \in \mathsf{M}_{n\times n}(F)$ then A is similar to A.

To see that A is similar to A we must find a matrix $Q \in \mathsf{M}_{n\times n}(F)$ with $A = Q^{-1}AQ$. But $A = I^{-1}AI$ so $Q = I$ will work.

(ii) If $A, B \in \mathsf{M}_{n\times n}(F)$ and A is similar to B then B is similar to A.

Supposing that A is similar to B means that we have an invertible matrix $Q \in \mathsf{M}_{n\times n}(F)$ with $A = Q^{-1}BQ$. We want to prove that B is similar to A, so we must find a matrix $P \in \mathsf{M}_{n\times n}$ which is invertible and $B = P^{-1}AP$. Certainly thought $B = QAQ^{-1}$ so $P = Q^{-1}$ works.

(iii) If $A, B, C \in \mathsf{M}_{n\times n}(F)$ where A is similar to B and B is similar to C then A is similar to C.

Since A is similar to B and B is similar to C there are invertible matrices $Q, P \in \mathsf{M}_{n\times n}(F)$ with

$$A = Q^{-1}BQ, \qquad B = P^{-1}CP.$$

But then

$$A = Q^{-1}BQ = Q^{-1}(P^{-1}AP)Q = (Q^{-1}P^{-1})A(PQ) = (PQ)^{-1}A(PQ)$$

where the last equality follows from the fact that $(PQ) = Q^{-1}P^{-1}$ (this is true by Exercise 4 in Section 2.4).

So the matrix $T = PQ$ satisfies $A = T^{-1}CT$ so that A is similar to A.

§2.5#12* **Theorem.** *Let $A \in \mathsf{M}_{n\times n}(F)$ and let $\gamma = \{v_1, \ldots, v_n\}$ be an ordered basis for F^n. Then $[L_A]_\gamma = Q^{-1}AQ$ where Q is the $n \times n$ matrix whose jth column is the jth vector of γ.*

Proof. First let $\beta = \{e_1, \ldots, e_n\}$ be the standard ordered basis for F^n, specifically e_i is the vector with a 1 in the ith coordinate and 0 everywhere else.

Then since $L_A(e_i)$ is the ith column of A, we have that $[L_A]_\beta = A$.

Then Theorem 2.23 $[L_A]\gamma = Q^{-1}[L_A]_\beta Q = Q^{-1}AQ$ where Q is the change of coordinates matrix that changes γ-coordinates into β-coordinates. So the jth column of Q is precisely $[v_j]\beta$.

The claim will follow if we can show that $[v_j]\beta = v_j$. In fact if $w \in F^n$ is any vector then $[w]_\beta = w$. To see this we note that there are $w_1, \ldots, w_n \in F$ with

$$w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \sum_{k=1}^{n} w_k e_k$$

so by definition $[w]_\beta = w$.

This means that the jth column of Q is $[v_j]_\beta = v_j$ which is the jth vector of γ. □