Homework Notes — Week 2

Math 24 — Spring 2014

§1.4:12* Remember to use theorems (and to correctly reference them). Proving everything from scratch every time takes a lot of time and effort!

In this case, the inclusion $\mathsf{W} \subseteq \mathrm{span}(\mathsf{W})$ is an immediate consequence of Theorem 1.5. The reverse inclusion $\mathrm{span}(\mathsf{W}) \subseteq \mathrm{span}(\mathsf{W})$ says, after unpacking the definition of $\mathrm{span}(\mathsf{W})$, that a the subspace W is closed under linear combinations. Stated this way, this is a consequence of Theorem 1.3.

A very careful proof of that a subspace W is closed under linear combinations would proceed by induction on the length $n \geq 1$ of the linear combination

$$a_1 w_1 + a_2 w_2 + \cdots + a_n w_n.$$

Base Case $(n = 1)$**.** If $w_1 \in \mathsf{W}$ and a_1 is any scalar, then $a_1 w_1 \in \mathsf{W}$ since W is closed under scalar multiplication by Theorem 1.3.

Induction Step $(n \to n + 1)$**.** Suppose we know that W is closed under linear combinations of length n. (This is the Induction Hypothesis.) Suppose we have vectors $w_1, \dots, w_n, w_{n+1}$ from W and scalars $a_1, \dots, a_n, a_{n+1}$. We want to show that

$$a_1 w_1 + \cdots + a_n w_n + a_{n+1} w_{n+1}$$

is in W. By the induction hypothesis, we know that

$$w = a_1 w_1 + \cdots + a_n w_n$$

is in W. We also know that $a_{n+1} w_{n+1}$ is in W by Theorem 1.3(c). By Theorem 1.3(b), we see that $w + a_{n+1} w_{n+1}$ is in W. Since

$$w + a_{n+1} w_{n+1} = a_1 w_1 + \cdots + a_n w_n + a_{n+1} w_{n+1},$$

we see that this linear combination of length $n + 1$ is indeed in W.

Because $w_1, \dots, w_n, w_{n+1}$ were arbitrary elements of W and $a_1, \dots, a_n, a_{n+1}$ were arbitrary scalars, we conclude that W is closed under linear combinations of length $n + 1$.

By the principle of mathematical induction, it follows that W is closed under linear combinations of any length.

§1.5:17* The most common approach to this problem was a proof by induction on n. Another way to solve this problem is to use Theorem 1 from the April 4 worksheet, observing that the i-th column couldn't be a linear combination of previous columns since its i-th entry is nonzero but the i-th entry is zero in all previous columns is zero because the matrix is upper triangular.

§1.6:11* We can split this into two theorems.

Theorem 1. *If $\{u, v\}$ is a two element basis for V and a is a nonzero scalar, then $\{u + v, au\}$ is also a basis for V.*

Proof. We first show that $\{u+v, au\}$ generates V. Given any $x \in \mathsf{V}$, there are scalars x_1, x_2 such that $x = x_1u + x_2v$. Then

$$x = x_2(u + v) + (x_1 - x_2)u = x_2(u + v) + \frac{x_1 - x_2}{a}(au)$$

shows that x is indeed a linear combination of the set $\{u + v, au\}$.

We now show that $\{u + v, au\}$ is linearly independent. Given any scalars x_1, x_2,

$$x_1(u + v) + x_2(au) = (x_1 + ax_2)u + x_1v.$$

Since $u \neq v$ and $\{u, v\}$ is linearly independent, we conclude that if $x_1(u+v)+x_2(au) = 0$ then $x_1 = 0$ and $x_1 + ax_2 = 0$. Since $a \neq 0$, the only possibility in this case is that $x_1 = x_2 = 0$. Therefore, $\{u + v, au\}$ is linearly independent.

Since $\{u + v, au\}$ is linearly independent and generates V, it is a basis for V. $\square$

Theorem 2. *If $\{u, v\}$ is a two element basis for V and a, b are a nonzero scalars, then $\{au, bv\}$ is also a basis for V.*

Proof. By Theorem 1.8, it is sufficient to show that every element of V has a *unique* representation as a linear combination of $\{au, bv\}$.

Since $\{u, v\}$ is a basis for V, we know that for every $x \in \mathsf{V}$, there are unique scalars x_1, x_2 such that $x = x_1u + x_2v$. It follows that

$$x = \frac{x_1}{a}(au) + \frac{x_2}{b}(bv)$$

is a linear combination of $\{au, bv\}$. Furthermore, this is the unique such linear combination for if

$$x = y_1(au) + y_2(bv) = (ay_1)u + (by_2)v,$$

then we must have $ay_1 = x_1$ and $by_2 = x_2$.

It follows from Theorem 1.8 that $\{au, bv\}$ is indeed a basis for V. $\square$

§1.6:14 When you asked to find a basis for a space W, it is necessary to justify that your proposed basis is indeed a basis for W. In this case, the dimensions of W_1 and W_2 are not known in advance, so we have to prove that our proposed basis does span the given space and that it is linearly independent.

The vectors in W_1 are precisely those $(a_1, a_2, a_3, a_4, a_5) \in F^5$ such that $a_1 = a_3 + a_4$. Each such vector is of the form

$$\begin{pmatrix} a_3 + a_4 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + a_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + a_5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The vectors $\{e_2, e_1 + e_3, e_1 + e_4, e_5\}$ are easily checked to be linearly independent. Indeed,

$$0 = pe_2 + q(e_1 + e_3) + r(e_1 + e_4) + se_5 = (p + r)e_1 + pe_2 + qe_3 + re_4 + se_5$$

can only happen if $p = q = r = s = 0$. Therefore $\{e_2, e_1 + e_3, e_1 + e_4, e_5\}$ is a basis for W_1 and it follows that $\dim(\mathsf{W}_1) = 4$.

The vectors in W_1 are precisely those $(a_1, a_2, a_3, a_4, a_5) \in F^5$ such that $a_1 = -a_5, a_2 = a_4, a_3 = a_4$ Each such vector is of the form

$$\begin{pmatrix} -a_5 \\ a_4 \\ a_4 \\ a_4 \\ a_5 \end{pmatrix} = a_4 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + a_5 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

The vectors in $\{e_2 + e_3 + e_4, -e_1 + e_5\}$ are easily checked to be linearly independent. Indeed,

$$0 = p(e_2 + e_3 + e_4) + q(-e_1 + e_5) = -qe_1 + pe_2 + pe_3 + pe_4 + qe_5$$

can only happen if $p = q = 0$. Therefore $\{e_2 + e_3 + e_4, -e_1 + e_5\}$ is a basis for W_2 and it follows that $\dim(\mathsf{W}_2) = 2$.

§1.6:16 I claim that the upper triangular $n \times n$ matrices have basis

$$\{E^{ij} : 1 \leq i \leq j \leq n\},$$

where E^{ij} are defined in Example 3 of Section 1.6 (where $m = n$).

From Example 3 in Section 1.6, we know that the larger set $\{E^{ij} : 1 \leq i, j \leq n\}$ is a basis for $\mathsf{M}_{n\times n}(F)$. Since every subset of a linearly independent set is linearly independent, our proposed basis is linearly independent.

To see that it generates all upper triangular $n \times n$ matrices, first note that if A is any $n \times n$ matrix, then

$$A = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} E^{ij}.$$

If A is upper triangular, $a_{ij} = 0$ when $i < j$ and so

$$A = \sum_{i=1}^{n} \sum_{j=i}^{n} a_{ij} E^{ij}.$$

This shows that every upper triangular matrix is a linear combination of elements from our proposed basis. Since every element of our proposed basis is upper triangular, we conclude that the span of our proposed basis is indeed the space of upper triangular matrices.

Our basis contains n elements with $i = 1$, $n - 1$ elements with $i = 2$, …, 2 elements with $i = n - 1$, and 1 element with $i = n$. So the dimension of the space of upper triangular matrices is

$$n + (n-1) + \cdots + 2 + 1 = \frac{n(n+1)}{2}.$$