Math 24
Spring 2012
Sample Homework Solutions
Week 9

Section 6.3

(2.) Find a vector y such that $g(x) = \langle x, y \rangle$ for all $x \in V$.

(b) $V = \mathbb{C}^2$, and $g(z_1, z_2) = z_1 - 2z_2$.

$y = (1, -2)$.

(3.) Evaluate T^* at the given vector in V.

(b) $V = \mathbb{C}^2$, $T(z_1, z_2) = (2z_1 + iz_2, (1-i)z_1)$, $x = (3-i, 1+2i)$.

$T^*(z_1, z_2) = (2z_1 + (i+1)z_2, (-i)z_1)$; $T^*(3-i, 1+2i) = (5+i, -1-3i)$.

(12.) Let V be an inner product space, and T a linear operator on T. Prove:

(a) $R(T^*)^\perp = N(T)$.

Suppose $w \in R(T^*)$ and $v \in V$ are arbitrary. By assumption, we can write $w = T^*(u)$ for some $u \in V$. Then, by the definition of T^*, we have

$$\langle v, w \rangle = \langle v, T^*(u) \rangle = \langle T(v), u \rangle .$$

If $v \in N(T)$, then $T(v) = 0$, and so, by the above equation, $\langle v, w \rangle = 0$ for all $w \in R(T^*)$. That is, $v \in R(T^*)^\perp$. This shows $N(T) \subseteq R(T^*)^\perp$.

Conversely, suppose that $v \in R(T^*)^\perp$. Then $\langle v, w \rangle = 0$ for all $w \in R(T^*)$, and so, by the above equation, $\langle T(v), u \rangle = 0$ for all $u \in V$. This can only happen if $T(v) = 0$, that is, $v \in N(T)$. This shows $R(T^*)^\perp \subseteq N(T)$.

(b) If V is finite-dimensional, then $R(T^*) = N(T)^\perp$.

By part (a), all vectors in $R(T^*)$ are orthogonal to all vectors in $N(T)$, and so $R(T^*) \subseteq N(T)^\perp$.

By Theorem 6.7(c), and part (a), $dim(R(T^*)) = V - dim(R(T^*)^\perp) = V - dim(N(T))$; by Theorem 6.7(c), $dim(N(T)^\perp = V - dim(N(T))$.

Now we have $R(T^*) \subseteq N(T)^\perp$, and $R(T^*)$ and $N(T)^\perp$ have the same finite dimension. Therefore, they must be equal.

(20.) Use the least squares approximation to find the best fit for this set of data with both a linear function and a quadratic function; compute the error E in both cases.

(a) $\{(-3, 9), (-2, 6), (0, 2), (1, 1)\}$.

For a linear function, we want the function $y = ct + d$ that best fits this data; that is, we want the closest possible approximation $x = x_0$ to a solution to the equation $Ax = y$:

$$\begin{pmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix}.$$

The error will be $E = ||y - Ax_0||^2$.

By our general method, our solution is $x_0 = (A^*A)^{-1}A^*y$. Since we are working over $\mathbb{R}$, $A^* = A^t$, and we can compute $A^*A = \begin{pmatrix} 14 & 4 \\ 4 & 4 \end{pmatrix}$, $(A^*A)^{-1} = \frac{1}{20}\begin{pmatrix} 2 & 2 \\ 2 & 7 \end{pmatrix}$,

$$\begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 2 & 2 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} -3 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2.5 \end{pmatrix},$$

our best linear approximation to this data is $y = -2t + 2.5$, and the error is $\left\| \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 2.5 \end{pmatrix} \right\|^2 = 1.$

For a quadratic function, our equation $Ax = y$ is

$$\begin{pmatrix} 9 & -3 & 1 \\ 4 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix}.$$

The best quadratic approximation is $y = \frac{1}{3}t^2 - \frac{4}{3}t + 2$ and the error is 0.

Section 6.4

(2.) For each linear operator T, determine whether T is normal, self-adjoint, or neither. If possible, find an orthonormal basis of eigenvectors, and give the corresponding eigenvalues.

(b) $V = \mathbb{R}^2$ and $T(a, b) = (2a - 2b, -2a + 5b)$.

The matrix of T in the standard basis is $\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$, which is self-adjoint, so T is self-adjoint. We compute the eigenvalues, 1 and 6, and corresponding eigenvectors, $(2, 1)$ and $(1, -2)$ respectively, and check that they are in fact orthogonal. An orthonormal basis of eigenvectors is $\left\{ \frac{\sqrt{5}}{5}(2, 1), \frac{\sqrt{5}}{5}(1, -2) \right\}$.

(d) $V = P_2(\mathbb{R})$ and $T(f) = f'$; the inner product is $\langle f, g \rangle = \int_0^1 f(t)g(t)\, dt$.

First, we use Gram-Schmidt to find an orthonormal basis relative to this inner product, and get $\beta = \{1, 2\sqrt{3}x - \sqrt{3}, 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}\}$. The matrix of T in this basis is $\begin{pmatrix} 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{15} \\ 0 & 0 & 0 \end{pmatrix}$, which we can check is neither self-adjoint nor normal.

(f) $V = M_{2\times 2}(\mathbb{R})$ and $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$.

The matrix of T in the standard basis is $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, which is self-adjoint. The eigenvalues are 1, with eigenvectors $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, and

-1, with eigenvectors $\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$. An orthonormal basis of eigenvectors is $\left\{ \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \right\}$

(9.) Let T be a normal operator on a finite-dimensional inner product space V. Prove that $N(T) = N(T^*)$ and $R(T) = R(T^*)$.

By Theorem 6.15(a), $||T(x)|| = ||T^*(x)||$ for all $x \in V$. Therefore, we have

$$x \in N(T) \iff T(x) = 0 \iff ||T(x)|| = 0 \iff ||T^*(x)|| = 0 \iff T^*(x) = 0 \iff x \in N(T^*).$$

Therefore $N(T) = N(T^*)$.

By 6.3 exercise (12), $R(T^*) = N(T)^\perp$. Since if T is normal, T^* is also, 6.3 exercise (12) applies to T^*, and we have $R(T) = R(T^{**}) = N(T^*)^\perp$. Since $N(T) = N(T^*)$, we have $R(T) = N(T^*)^\perp = N(T)^\perp = R(T^*)$.

Section 6.5

(2.) For each matrix A, find a unitary or orthogonal matrix P and a diagonal matrix D such that $P^*AP = D$.

(b) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The eigenvalues of A are i, with eigenvector $(i, 1)$, and $-i$, with eigenvector $(1, i)$. An orthonormal basis of eigenvectors is $\left\{ (\frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2}), (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}i) \right\}$. $D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, and $P = \begin{pmatrix} \frac{\sqrt{2}}{2}i & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}i \end{pmatrix}$.

(5.) Are these matrices unitarily equivalent?

(b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$.

No. Unitarily equivalent matrices must be similar, and similar matrices have the same determinant; these matrices do not have the same determinant.

Section 6.6

(2.) Compute the matrix of T in the standard basis β, where T is the orthogonal projection of V on W, if $V = \mathbb{R}^2$ and $W = span(\{(1,2)\})$, and if $V = \mathbb{R}^3$ and $W = span(\{(1,0,1)\})$

To find $T(e_i)$ where $W = span(\{w\})$, find the orthogonal projection of e_i onto the vector w, which is $\dfrac{\langle e_i, w\rangle}{\langle w, w\rangle} w$.

In the first case, $[T]_\beta = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}$.

In the second case, $[T]_\beta = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$.

(3.) For the corresponding matrix A in 6.5 exercise (2), verify that L_A has a spectral decomposition, for each eigenvalue explicitly define the projection onto the corresponding eigenspace, and use the spectral theorem to verify the result.

(b) We can check to see that $AA^* = A^*A$, so A is normal, and has a spectral decomposition (over $\mathbb{C}$). An orthonormal basis of eigenvectors is $\left\{(\frac{\sqrt{2}}{2}i,\ \frac{\sqrt{2}}{2}), (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}i)\right\}$. The projection of $(1,0)$ onto the eigenvector $(\frac{\sqrt{2}}{2}i,\ \frac{\sqrt{2}}{2})$ is $\left\langle (1,0), (\frac{\sqrt{2}}{2}i,\ \frac{\sqrt{2}}{2}) \right\rangle (\frac{\sqrt{2}}{2}i,\ \frac{\sqrt{2}}{2}) = (\frac{1}{2}, -\frac{1}{2}i)$ and the projection of $(0,1)$ onto the eigenvector $(\frac{\sqrt{2}}{2}i,\ \frac{\sqrt{2}}{2})$ is $\left\langle (0,1), (\frac{\sqrt{2}}{2}i,\ \frac{\sqrt{2}}{2}) \right\rangle (\frac{\sqrt{2}}{2}i,\ \frac{\sqrt{2}}{2}) = (\frac{1}{2}i, \frac{1}{2})$, so the matrix of the orthogonal projection onto E_i in the standard basis is $\begin{pmatrix} \frac{1}{2} & \frac{1}{2}i \\ -\frac{1}{2}i & \frac{1}{2} \end{pmatrix}$, and the projection is $T_i(a,b) = (\frac{1}{2}a + \frac{1}{2}ib, -\frac{1}{2}ia + \frac{1}{2}b)$.

Similarly, the projection onto E_{-i} is $T_{-i}(a,b) = T_i(a,b) = (\frac{1}{2}a - \frac{1}{2}ib, \frac{1}{2}ia + \frac{1}{2}b)$.

To verify this using the spectral theorem, we need to check that $L_A = iT_i + (-i)T_{-i}$, or that $TL_A(a,b) = i(\frac{1}{2}a + \frac{1}{2}ib, -\frac{1}{2}ia + \frac{1}{2}b) + (-i)(\frac{1}{2}a - \frac{1}{2}ib, \frac{1}{2}ia + \frac{1}{2}b)$. This is true.