Math 24
Spring 2012
Sample Homework Solutions
Week 4

Section 2.3

(3) Let $g(x) = 3 + x$. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $U : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations respectively defined by

$$T(f(x)) = f'(x)g(x) + 2f(x)$$

$$U(a + bx + cx^2) = (a + b,\ c,\ a - b).$$

Let β and γ be the standard ordered bases of $P_3(\mathbb{R})$ and $\mathbb{R}^3$ respectively.

(a) Compute $[U]_\beta^\gamma$, $[T]_\beta$ and $[UT]_\beta^\gamma$ directly. Then use Theorem 2.11 to verify your result.

To do the direct computations, we need to see that $T(1) = 2$, $T(x) = 3+3x$, $T(x^2) = 6x+4x^2$, $U(1) = (1, 0, 1)$, $U(x) = (1, 0, -1)$, $U(x^2) = (0, 1, 0)$, $UT(1) = U(2) = (2, 0, 2)$, $UT(x) = U(3 + 3x) = (6, 0, 0)$, $UT(x^2) = U(6x + 4x^2) = (6, 4, -6)$. This tells us that

$$[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix} \qquad [U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \qquad [UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$$

To use Theorem 2.11 to verify this, we must check that $[UT]_\beta^\gamma = [U]_\beta^\gamma [T]_\beta$, which is true.

(b) Let $h(x) = 3 - 2x + x^2$. Compute $[h(x)]_\beta$ and $[U(h(x))]_\gamma$. Then use $[U]_\beta^\gamma$ from (a) and Theorem 2.14 to verify your result.

$[h(x)]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$, $U(h(x)) = (1, 1, 5)$, and so $[U(h(x))]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$. To use Theorem 2.14 to verify this, we must check that $[U]_\beta^\gamma [h(x)]_\beta = [U(h(x))]_\gamma$, which is true.

(4) For each part, let T be as defined in the corresponding part of Section 2.2, exercise (5), and use Theorem 2.14 to compute the given vector:

(b) $[T(f(x))]_\alpha$, where $f(x) = 4 - 6x + 3x^2$.

In 2.2(5), where α and β are the standard ordered bases of $M_{2\times 2}(\mathbb{R})$ and $P_2(\mathbb{R})$ and $T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$, we showed that $[T]_\beta^\alpha = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Therefore

$$[T(f(x))]_\alpha = [T]_\beta^\alpha [f(x)]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ -6 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 0 \\ 6 \end{pmatrix}.$$

(d) $[T(f(x))]_\gamma$, where $f(x) = 6 - x + 2x^2$.

In 2.2(5), where γ and β are the standard ordered bases of $\mathbb{R}^1$ and $P_2(\mathbb{R})$ and $T(f(x)) = f(2)$, we showed that $[T]_\beta^\gamma = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$. Therefore

$$[T(f(x))]_\gamma = [T]_\beta^\gamma [f(x)]_\beta = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix} = 12.$$

(12)(a) Let V, W, and Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear. Prove that if UT is one-to-one, then T is one-to-one. Must U also be one-to-one?

We use the fact that a linear transformation is one-to-one if and only if its null space is $\{0\}$, and prove the contrapositive.

Suppose that T is not one-to-one. Then $N(T) \neq \{0\}$; that is, there is some nonzero vector $v \in N(T)$, so $T(v) = 0$. Then $UT(v) = U(T(v)) = U(0) = 0$, which shows $v \in N(UT)$. Since v is nonzero, $N(UT) \neq \{0\}$, and so UT is not one-to-one.

It is possible for UT to be one-to-one but U not to be one-to-one. For example, let $T : \mathbb{R}^2 \to \mathbb{R}^3$ and $U : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(x, y) = (x, y, 0)$

and $U(x, y, z) = (x, y)$. Then $UT(x, y) = U(x, y, 0) = (x, y)$, so UT is one-to-one. However, U is not one-to-one; the null space of U is the z-axis.

Section 2.4

(4) Let A and B be $n \times n$ invertible matrices. Prove that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Using the associativity of matrix multiplication,

$$\big(AB\big)\big(B^{-1}A^{-1}\big) = A\big(BB^{-1}\big)A^{-1} = AA^{-1} = I$$

$$\big(B^{-1}A^{-1}\big)\big(AB\big) = B^{-1}\big(A^{-1}A\big)B = B^{-1}B = I.$$

Since $B^{-1}A^{-1}$ multiplied by AB in either order gives the identity matrix, by the definition of the inverse of a matrix, $B^{-1}A^{-1} = (AB)^{-1}$. Since AB has an inverse, AB is invertible.

(6) Prove that if A is invertible and $AB = 0$ then $B = 0$. Here 0 denotes the zero matrix, all of whose entries are zero.

Since every entry of 0 is zero, the product of 0 with any matrix (of the right shape to be multiplied by it) is again a zero matrix. We multiply both sids of $AB = 0$ on the left by A^{-1} to get

$$A^{-1}AB = A^{-1}0$$

$$B = 0.$$

(15) Let V and W be finite-dimensional vector spaces and $T : V \to W$ be a linear transformation. Suppose that β is a basis for V. Prove that T is an isomorphism if and only if $T(\beta)$ is a basis for W

Let $\beta = \{v_1, v_2, \ldots, v_n\}$ and $T(v_i) = w_i$.

By Theorem 2.2, page 68, $T(\beta) = \{w_1, w_2, \ldots, w_n\}$ spans $R(T)$. Since T is onto if and only if $R(T) = W$, this tells us that T is onto if and only if $T(\beta)$ spans W.

Now we argue that T is one-to-one if and only if $T(\beta)$ is linearly independent. This will finish the proof, because we will have that T is an isomorphism iff T is both one-to-one and onto, iff $T(\beta)$ both is linearly independent and spans W, iff $T(\beta)$ is a basis for W.

First, suppose that $T(\beta)$ is not linearly independent. Therefore, for some a_i not all zero, we have

$$a_1 w_1 + a_2 w_2 + \cdots + a_n w_n = 0,$$

and

$$T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n) = a_1 w_1 + a_2 w_2 + \cdots + a_n w_n = 0.$$

Since β is linearly independent, $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \neq 0$, so $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$ is a nonzero element of $N(T)$, showing $N(T) \neq \{0\}$, and T is not one-to-one.

Conversely, suppose that T is not one-to-one. Then there is some nonzero $v \in V$ such that $T(v) = 0$. Because β is a basis, we can write $v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$; clearly, not all the a_i are zero. Now we have

$$0 = T(v) = T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n) = a_1 w_1 + a_2 w_2 + \cdots + a_n w_n,$$

which shows that $T(\beta)$ is not linearly independent.

Section 2.5

(5) Let T be the linear operator on $P_1(\mathbb{R})$ defined by $T(p(x)) = p'(x)$, the derivative of p. Let $\beta = \{1, x\}$ and $\beta' = \{1 + x, 1 - x\}$. Use Theorem 2.21 and the fact that

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

to find $[T]_{\beta'}$.

The matrix $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ changes from β'-coordinates into β-coordinates, since its columns are the β-coordinates of the elements of β'. The derivative operator T has matrix in the standard basis $[T]_\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; the columns of this matrix are the β-coordinates of $T(1)$ and $T(x)$. Therefore we can write

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

We can check that the columns of this matrix are the β'-coordinates of $T(1+x) = 1$ and $T(1-x) = -1$.

(6) For each matrix A and ordered basis β find $[L_A]_\beta$. Also, find an invertible matrix Q such that $[L_A]_\beta = Q^{-1}AQ$.

Discussion of this problem:

If A is an $n \times n$ matrix over a field F, to compute L_A of an element x of F^n, you simply write x as a column vector and multiply on the left by A. This is the definition of L_A. Of course, the product Ax also gives you $L_A(x)$ written as a column vector.

Now, if α is the standard basis for F^n, then $[x]_\alpha$ is just x written as a column vector Therefore, by the definition of L_A,

$$[L_A(x)]_\alpha = A[x]_\alpha.$$

This means

$$[L_A]_\alpha = A.$$

In each part, the matrix Q must be the matrix that changes from β-coordinates into standard coordinates; its columns are the standard coordinates of the vectors in β. Then we will have

$$[L_A]_\beta = Q^{-1}[L_A]_\alpha Q = Q^{-1}AQ.$$

We might not compute $[L_A]_\beta$ this way, since it would involve inverting Q. Instead we can use the fact that its columns are the β-coordinates of Av for $v \in \beta$. (We could also find Q^{-1}, since it converts from standard to β-coordinates, so its columns are the β-coordinates of the standard basis vectors.)

(a) $A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

$L_A(1,1) = (4,2) = 6(1,1) - 2(1,2)$ and $L_A(1,2) = (7,3) = 11(1,1) - 4(1,2)$. Therefore

$$[L_A]_\beta = \begin{pmatrix} 6 & 11 \\ -2 & -4 \end{pmatrix}.$$

(c) $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$[L_A]_\beta = \begin{pmatrix} 2 & 2 & 2 \\ -2 & -3 & -4 \\ 1 & 1 & 2 \end{pmatrix}$$

(7) In $\mathbb{R}^2$, let L be the line $y = mx$ where $m \neq 0$. Find an expression for $T(x, y)$.

In both parts of this problem, we can begin by choosing an ordered basis $\beta = \{\vec{v}_1, \vec{v}_2\}$, where $\vec{v}_1$ lies in L and $\vec{v}_2$ is perpendicular to L. Then the reflection of $\vec{v}_1$ about L, and the projection of $\vec{v}_1$ onto L, are both $\vec{v}_1$, while the reflection of $\vec{v}_2$ about L is $-\vec{v}_2$, and the projection of $\vec{v}_2$ onto L is $\vec{0}$.

Two points on L are $(0, 0)$ and $(1, m)$, so we can take $\vec{v}_1 = (1, m)$ and $\vec{v}_2 = (-m, 1)$.

This tells us that the matrix that changes from β-coordinates to standard coordinates is $Q = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}$. Either by solving a system of linear equations or by using the formula for the inverse of a 2×2 matrix we saw in class, we see that the matrix for changing from standard coordinates to β-coordinates is $Q^{-1} = \begin{pmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ -\frac{m}{1+m^2} & \frac{1}{1+m^2} \end{pmatrix}$.

Therefore, if α is the standard basis for $\mathbb{R}^2$, we can write $[T]_\alpha = Q[T]_\beta Q^{-1}$, and use $[T(x, y)]_\alpha = [T]_\alpha[(x, y)]_\alpha = [T]_\alpha \begin{pmatrix} x \\ y \end{pmatrix}$ to find $T(x, y)$.

The computations are left for you.

(a) T is the reflection of $\mathbb{R}^2$ about L.

In this case $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(b) T is the projection on L along the line perpendicular to L.

In this case $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.