Math 24
Spring 2012
Sample Homework Solutions
Week 3

In-class problems from Monday, April 9:

(2) Give examples of pairs of subspaces W_1 and W_2 of $\mathbb{R}^3$, neither of which is contained in the other, such that:

(a) $W_1 + W_2 \neq \mathbb{R}^3$. In your example, what is $W_1 + W_2$?

(b) $W_1 + W_2 = \mathbb{R}^3$, but $\mathbb{R}^3$ is not the direct sum of W_1 and W_2. In your example, what is $W_1 \cap W_2$?

(c) $\mathbb{R}^3$ is the direct sum of W_1 and W_2.

For (a) we can take W_1 to be the x-axis and W_2 to be the y-axis. In this case, $W_1 + W_2$ is the xy-plane.

For (b) we can take W_1 to be the xz-plane and W_2 to be the yz-plane. In this case, $W_1 \cap W_2$ is the z-axis.

For (c) we can take W_1 to be the x-axis and W_2 to be the yz-plane.

(5) Express $P(\mathbb{R})$ as the direct sum of two nonzero subspaces in two ways.

(a) One of the subspaces has finite dimension.

(b) Both of the subspaces are infinite-dimensional.

For (a) we can take one subspace to be $P_0(\mathbb{R})$ (all constants), and the other to be all polynomials whose constant term is 0.

For (b) we can take one subspace to be all linear combinations of even powers of x (including x^0, that is, 1), and the other to be all linear combinations of odd powers of x.

(6) Prove the conjecture you made in problem (3). Hint: A basis $\{x_1, \ldots, x_k\}$ for $W_1 \cap W_2$ can be extended to a basis $\{x_1, \ldots, x_k, y_1, \ldots, y_n\}$ for W_1. It can also be extended to a basis $\{x_1, \ldots, x_k, z_1, \ldots, z_m\}$ for W_2. For homework, you might want to verify your conjecture by looking at problem 29(a) of section 1.6 of the textbook. Please make a conjecture yourself first, though.

We want to prove that if a finite-dimensional vector space V is the direct sum of subspaces W_1 and W_2, then

$$dim(V) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2).$$

As suggested in the hint, let $\{x_1, \ldots, x_k\}$ be a basis for $W_1 \cap W_2$. Since this is a linearly independent subset of W_1, it can be extended to a basis $\{x_1, \ldots, x_k, y_1, \ldots, y_n\}$ for W_1. Similarly, it can be extended to a basis $\{x_1, \ldots, x_k, z_1, \ldots, z_m\}$ for W_2.

We see from these bases that

$$dim(W_1) = k + n,\ dim(W_2) = k + m,\ dim(W_1 \cap W_2) = k,$$

so the number of elements in $S = \{x_1, \ldots, x_k, y_1, \ldots, y_n, z_1, \ldots, z_m\}$ is

$$n + m + k = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2).$$

Therefore we will prove the conjecture if we show that S is a basis for V.

First we show that S spans V. Because $V = W_1 + W_2$, every $v \in V$ can be written as a sum of elements of W_1 and W_2,

$$v = w_1 + w_2.$$

The elements $w_1 \in W_1$ and $w_2 \in W_2$ can be written as linear combinations of their respective basis elements,

$$w_1 = a_1x_1 + \cdots + a_kx_k + b_1y_1 + \cdots + b_ny_n$$

$$w_2 = c_1x_1 + \cdots + c_kx_k + d_1z_1 + \cdots + d_mz_m.$$

Substituting the right hand sides of these two equations for w_1 and w_2 in $v = w_1 + w_2$, we get

$$v = a_1x_1 + \cdots + a_kx_k + b_1y_1 + \cdots + b_ny_n + c_1x_1 + \cdots + c_kx_k + d_1z_1 + \cdots + d_mz_m.$$

This shows every $v \in V$ can be written as a linear combination of elements of S, so S spans V.

Now we show that S is linearly independent. Suppose some linear combination from S equals zero,

$$a_1x_1 + \cdots + a_kx_k + b_1y_1 + \cdots + b_ny_n + c_1z_1 + \cdots + c_mz_m = 0.$$

We must show all the coefficients are zero.

We can rewrite this equation as

$$a_1x_1 + \cdots + a_kx_k + b_1y_1 + \cdots + b_ny_n = -c_1z_1 - \cdots - c_mz_m.$$

On the left hand side we have a linear combination of basis elements from W_1, therefore, an element of W_1; on the right hand side we have a linear combination of basis elements from W_2, therefore, an element of W_2; hence the vector given by these linear combinations is in the intersection $W_1 \cap W_2$. But V is the direct sum of $W_1 + W_2$, so the only element of $W_1 \cap W_2$ is zero:

$$a_1x_1 + \cdots + a_kx_k + b_1y_1 + \cdots + b_ny_n = 0.$$

$$-c_1z_1 - \cdots - c_mz_m = 0.$$

This first equation tells us a linear combination of the x_i and y_j equals zero. Since the x_i and y_j form a basis for W_1, they are linearly independent, so all the coefficients a_i and b_j must be zero. Similarly, the second equation tells us a linear combination of elements from a basis for W_2 equals zero, so all the coefficients c_ℓ must be zero. This is what we needed to show.

This completes the proof.

Section 2.1

(4) Prove that T is a linear transformation; find bases for $N(T)$ and $R(T)$; verify the dimension theorem; determine whether T is one-to-one and whether T is onto.

$$T : M_{2\times 3}(F) \to M_{2\times 2}(F)$$

$$T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}.$$

The null space $N(T)$ consists of all matrices such that

$$2a_{11} - a_{12} = 0$$

$$a_{13} + 2a_{12} = 0.$$

We can rewrite these equations as

$$a_{12} = 2a_{11}$$

$$a_{13} = -4a_{11}.$$

Choosing parameters a, b, c, and d for a_{11}, a_{21}, a_{22} and a_{23}, we see that $N(T)$ consists of all matrices of the form

$$\begin{pmatrix} a & 2a & -4a \\ b & c & d \end{pmatrix},$$

which has a basis,

$$\left\{ \begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \right\}.$$

The range $R(T)$ is spanned by the images of the basis vectors of the domain, that is, by

$$\left\{ \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \right\}.$$

Reducing this to a linearly independent set with the same span gives a basis for $R(T)$,

$$\left\{ \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \right\}.$$

From these bases we see the nullity of T is 4 and the rank of T is 2. Since the dimension of the domain of T is 6, and $2+4=6$, we have verified the Dimension Theorem.

Because the nullity of T is not zero, T is not one-to-one. Because the rank of T is not the dimension of the codomain, T is not onto.

(10) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, and let $T(1,0) = (1,4)$ and $T(1,1) = (2,5)$. What is $T(2,3)$? Is T one-to-one.

It is useful to note that

$$T(0,1) = T((1,1)-(1,0)) = T(1,1) - T(1,0) = (2,5) - (1,4) = (1,1).$$

Therefore,

$$T(2,3) = T(2(1,0)+3(0,1)) = 2T(1,0)+3T(0,1) = 2(1,4)+3(1,1) = (5,11).$$

Since the images of the basis vectors $(1,0)$ and $(0,1)$ are linearly independent, T must be one-to-one.

(15) $T : P(\mathbb{R}) \to P(\mathbb{R})$ is defined by

$$T(f(x)) = \int_0^x f(t)\,dt.$$

Prove that T is linear and one-to-one but not onto.

T is linear because we know from calculus that

$$\int_0^x (cf+g)(t)\,dt = c\int_0^x f(t)\,dt + \int_0^x g(t)\,dt.$$

T is one-to-one because if $T(f(x)) = T(g(x))$ then we have

$$\int_0^x f(t)\,dt = \int_0^x g(t)\,dt,$$

$$\int_0^x (f-g)(t)\,dt = 0,$$

and the only polynomial whose integral over every interval is zero is the zero function, so $f(x) = g(x)$.

T is not onto, because if $f(x)$ is any polynomial, the constant term of

$$T(f(x)) = \int_0^x f(t)\,dt$$

is zero, so 1 (for example) is not in the range of T.

(17) Let V and W be finite-dimensional vector spaces, and $T : V \to W$ be linear. Prove that if $dim(V) < dim(W)$ then V cannot be onto, and if $dim(V) > dim(W)$ then T cannot be one-to-one.

The dimension theorem tells us that

$$n(T) + r(T) = dim(V),$$

and $n(T)$, $r(T)$ and $dim(V)$ are all non-negative.

If $dim(V) < dim(W)$, then

$$r(T) = dim(V) - n(T) \leq dim(V) < dim(W),$$

and since $r(T) \neq dim(W)$, T cannot be onto.

If $dim(V) > dim(W)$, then since $r(T)$ is the dimension of a subspace of W, we have $r(T) \leq dim(W) < dim(V)$, so

$$n(T) = dim(V) - r(T) < 0,$$

and since $n(T) \neq 0$, T cannot be one-to-one.

(18) Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $N(T) = R(T)$.

Let $T(x, y) = (y, 0)$. Then the x-axis is the null space and the range of T.

Section 2.2

(3) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be the standard basis for $\mathbb{R}^2$ and $\gamma = \{(1,1,0),(0,1,1),(2,2,3)\}$. Compute $[T]_\beta^\gamma$. If $\alpha = \{(1,2),(2,3)\}$, compute $[T]_\alpha^\gamma$.

$$T(1,0) = (1,1,2) = \left(-\frac{1}{3}\right)(1,1,0) + (0)(0,1,1) + \left(\frac{2}{3}\right)(2,2,3)$$

$$T(0,1) = (-1,0,1) = (-1)(1,1,0) + (1)(0,1,1) + (0)(2,2,3)$$

$$[T]_\beta^\gamma = \begin{pmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{pmatrix}$$

$$T(1,2) = T(1,0) + 2T(0,1) =$$
$$\left(\left(-\frac{1}{3}\right)(1,1,0) + (0)(0,1,1) + \left(\frac{2}{3}\right)(2,2,3)\right) +$$
$$2\left((-1)(1,1,0) + (1)(0,1,1) + (0)(2,2,3)\right) =$$
$$\left(-\frac{7}{3}\right)(1,1,0) + (2)\,(0,1,1) + \left(\frac{2}{3}\right)(2,2,3).$$

Similarlly,

$$T(2,3) = \left(-\frac{11}{3}\right)(1,1,0) + (3)\,(0,1,1) + \left(\frac{4}{3}\right)(2,2,3).$$

$$[T]_\alpha^\gamma = \begin{pmatrix} -\frac{7}{3} & -\frac{11}{3} \\ 2 & 3 \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix}$$

(5) $\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$,
$\beta = \{1, x, x^2\}$
$\gamma = \{1\}$
(b) $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$

$T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$

$[T]_\beta^\alpha = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$

To see where the second column comes from, the second basis element of β is $f(x) = x$, so for this polynomial $f'(x) = 1$ and $f''(x) = 0$, so $T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$.

(d) $T : P_2(\mathbb{R}) \to \mathbb{R}$
$T(f(x)) = f(2)$
$[T]_\beta^\gamma = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$.

(e) $A = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix} = (1)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (-2)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (0)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + (4)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

$[A]_\alpha = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 4 \end{pmatrix}.$

(8) Let V be an n-dimensional vector space with an ordered basis β. Define $T : V \to F^n$ by $T(x) = [x]_\beta$. Prove that T is linear.

Let $\beta = \{x_1, x_2, \ldots x_n\}$.
Let v and w be arbitrary elements of V. To find their coordinates, write them as linear combinations of the basis elements,

$$v = a_1x_1 + a_2x_2 + \cdots + a_nx_n \qquad w = b_1x_1 + b_2x_2 + \cdots + b_nx_n.$$

Then their β-coordinates are the corresponding coefficients,

$$[v]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \qquad [w]_\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Now for any scalar c we have

$$cv + w = c(a_1x_1 + a_2x_2 + \cdots + a_nx_n) + (b_1x_1 + b_2x_2 + \cdots + b_nx_n) =$$

$$(ca_1 + b_1)x_1 + (ca_2 + b_2)x_2 + \cdots + (ca_n + b_n)x_n$$

so we have

$$[cv + w]_\beta = \begin{pmatrix} ca_1 + b_1 \\ ca_2 + b_2 \\ \vdots \\ ca_n + b_n \end{pmatrix}.$$

Therefore,

$$T(cv + w) = [cv + w]_\beta = \begin{pmatrix} ca_1 + b_1 \\ ca_2 + b_2 \\ \vdots \\ ca_n + b_n \end{pmatrix} = c \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} =$$

$$c[v]_\beta + [w]_\beta = cT(v) + T(w),$$

which shows that T is linear.