Math 24
Spring 2012
Sample Homework Solutions
Week 1

Section 1.2

(9) Prove Corollaries 1 and 2 to Theorem 1.1, and Theorem 1.2(c).

Corollary 1: The vector 0 described in (VS 3) is unique.

Proof: Suppose $0'$ is another additive identity. We must show $0 = 0'$. Because $0'$ is an additive identity, we have $0+0' = 0$, and by the commutativity of addition we have $0' + 0 = 0$. Because 0 is an additive identity we have $0 + 0 = 0$. By Theorem 1.1 we can "cancel out" the $+0$, and so $0' = 0$.

Corollary 2: The vector y described in (VS 4) is unique.

Proof: Let y be as in (VS 4), so $x + y = 0$, and let z be another such vector, so $x + z = 0$. We must show $y = z$.
By the commutativity of addition we have $y + x = 0 = z + x$, so by Theorem 1.1 we can "cancel out" the $+x$, and so $y = z$.

Theorem 1.2(c): In any vector space V over a field F, for each $a \in F$, we have $a0 = 0$.

Proof: Because 0 is an additive identity, $0 + 0 = 0$, and so $a(0 + 0) = a0$. Because multiplication by scalars distributes over addition of vectors, we have $a0 + a0 = a0$.
Because 0 is an additive identity and addition is commutative, $a0 = a0 + 0 = 0 + a0$, and putting this together with $a0 + a0 = a0$ we have $a0 + a0 = 0 + a0$.
Now, by Theorem 1.1, $a0 = 0$.

(18) Let $V = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{R}$, define

$$(a_1, a_1) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \text{ and } c(a_2, a_2) = (ca_1, ca_2).$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

No. $(1, 0) + (0, 0) = (1, 0)$ and $(0, 0) + (1, 0) = (2, 0)$, which shows that addition (defined in this way) is not commutative, so (VS 1) fails.

Section 1.3

(10) Prove that $W_1 = \{(a_1, \ldots, a_n) \in F^n \mid a_1 + \cdots + a_n = 0\}$ is a subspace of F^n, but $W_2 = \{(a_1, \ldots, a_n) \in F^n \mid a_1 + \cdots + a_n = 1\}$ is not.

Since the zero vector $(0, \ldots, 0)$ is in W_1 but not in W_2, we know W_2 cannot be a subspace. To show W_1 is, take any two vectors x and y in W_1 and a scalar c in F, and show that cu and $u + v$ are in W_1.

Let $u = (a_1, \ldots, a_n)$ and $v = (b_1, \ldots, b_n)$ be in W_1, so $a_1 + \cdots + a_n = 0$ and $b_1 + \cdots + b_n = 0$. Then we have $cu = (ca_1, \ldots, ca_n)$ and $u + v = (a_1 + b_1, \ldots, a_n + b_n)$.

$$ca_1 + \cdots + ca_n = c(a_1 + \cdots + a_n) = c(0) = 0;$$

$$(a_1 + b_1) + \cdots + (a_n + b_n) = (a_1 + \cdots + a_n) + (b_1 + \cdots + b_n) = 0 + 0 = 0;$$

this shows that cu and $u + v$ are in W_1.

(15) Is the set of all differentiable real-valued functions defined on $\mathbb{R}$ a subspace of $C(\mathbb{R})$? Justify your answer.

Yes. We know from calculus that a differentiable function must be continuous, so this set is a subset of $C(\mathbb{R})$. We also know from calculus that the sum of differentiable functions is differentiable, and a constant multiple of a differentiable function is differentiable. Therefore this set is closed under addition and multiplication by scalars; since it also contains the zero function, it is a subspace.

(19) Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

If $W_1 \subseteq W_2$, then $W_1 \cup W_2 = W_2$, so $W_1 \cup W_2$ is a subspace of V. In the same way, if $W_2 \subseteq W_1$ then $W_1 \cup W_2$ is a subspace of V.

Conversely, suppose $W_1 \cup W_2$ is a subspace of V. To show that either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, suppose not. Then there is $w_1 \in W_1, w_1 \notin W_2$ and there is $w_2 \in W_2, w_2 \notin W_1$. As $W_1 \cup W_2$ is a subspace, $w_1 + w_2 \in W_1 \cup W_1$, so $w_1 + w_2$ is either in W_1 or in W_2.

Without loss of generality, $w_1 + w_2 \in W_1$. Then, because W_1 is a subspace, $w_2 = (w_1 + w_2) - w_1 \in W_1$. This is a contradiction, since $w_2 \notin W_1$. Therefore we must have either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.