Math 24

Spring 2012

Wednesday, April 11

(1.) TRUE or FALSE? In these exercises, V and W are finite-dimensional vector spaces over the same field F, and T is a function from V to W.

a. If T is linear, then T preserves sums and scalar products. (T)

b. If $T(x+y) = T(x) + T(y)$, then T is linear. (F)

c. T is one-to-one if and only if the only vector x such that $T(x) = 0$ is $x = 0$. (F) (True if T is linear.)

d. If T is linear, then $T(0_V) = 0_W$. (T)

e. If T is linear, then $nullity(T) + rank(T) = dim(W)$. (F)

f. If T is linear, then T carries linearly independent subsets of V onto linearly independent subsets of W. (F)

g. If $T : V \to W$ and $U : V \to W$ are both linear and agree on a basis for V (meaning that if x is in the basis, $T(x) = U(x)$), then $T = U$. (T)

h. Given $x_1, x_2 \in V$ and $y_1, y_2 \in W$, there exists a linear transformation $T : V \to W$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$. (F)

i. Recall that we can consider $\mathbb{R}$ to be a vector space over itself. Any function $T : \mathbb{R} \to \mathbb{R}$ of the form $T(x) = mx + b$, where m and b are constants in $\mathbb{R}$, is linear. (F) (It's an *affine* function, the sum of a linear function and a constant function.)

j. The words "range," "image," and "codomain" all mean the same thing. (F)

k. If $T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ is linear and $N(T)$ is the subspace of diagonal matrices in $M_{2\times 2}(\mathbb{R})$, then T is not onto. (T)

(2.) Explain why we know that the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ is not linear.

Just for fun, we'll use a different argument for each example.

a. $T(a_1,\, a_2) = (1,\, a_2)$.
$T(0,0) \neq (0,0)$.

b. $T(a_1,\, a_2) = (a_1,\, (a_1)^2)$.
$2(T(1,0)) \neq T(2(1,0))$.

c. $T(a_1,\, a_2) = (\sin(a_1),\, 0)$.
The null space of T is not a subspace of $\mathbb{R}^2$. (It includes $(\pi, 0)$ but not $\frac{1}{2}(\pi, 0)$.)

d. $T(a_1,\, a_2) = (|a_1|,\, a_2)$.
The range of T is not a subspace of $\mathbb{R}^2$. (It includes $(1,1)$ but not $-(1,1)$.)

e. $T(a_1,\, a_2) = (a_1 + 1,\, a_2)$.
$T((0,0) + (0,0)) \neq T(0,0) + T(0,0)$.

(3.) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x,y,z) = (2x - y, 2y - z, 4x - z)$.

a. Find a basis for the null space of T.

We need to find a basis for the solution space for the system of linear equations

$$\begin{aligned} 2x - y &= 0 \\ 2y - z &= 0 \\ 4x - z &= 0. \end{aligned}$$

Gaussian elimination converts this system to

$$\begin{aligned} \boxed{x} - \frac{1}{4}z &= 0 \\ \boxed{y} - \frac{1}{2}z &= 0 \\ 0 &= 0. \end{aligned}$$

We use the first two equations to solve for x and y, and set z equal to a parameter, s:

$$(x,y,z) = \left(\frac{1}{4}s,\, \frac{1}{2}s,\, s\right) = s\left(\frac{1}{4},\, \frac{1}{2},\, 1\right).$$

The null space consists of all vectors of this form, and a basis is

$$\left\{\left(\frac{1}{4},\, \frac{1}{2},\, 1\right)\right\}.$$

b. Find a basis for the range of T.

Since the domain of T is spanned by the set $\{(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1)\}$, the range of T is spanned by the set

$$\{T(1, 0, 0),\ T(0, 1, 0),\ T(0, 0, 1)\} = \{(2, 0, 4),\ (-1, 2, 0),\ (0, -1, -1)\}.$$

Since this set spans the range, we can reduce it to a basis, by considering its elements one by one and eliminating any that is a linear combination of the earlier ones. This gives us the basis

$$\{(2, 0, 4),\ (-1, 2, 0)\}.$$

c. Find the nullity and rank of T. Verify the dimension theorem (in the case of T).

The dimension theorem tells us that

$$N(T) + R(T) = dim(domain(T)).$$

The nullity of T is the dimension of the null space, in this case $N(T) = 1$, the rank of T is the dimension of the range, in this case $R(T) = 2$, and in this case the domain of T is $\mathbb{R}^3$, so $dim(domain(T)) = 3$. It is true that

$$1 + 2 = 3,$$

which verifies the dimension theorem in this case.

d. Is T one-to-one? How can you tell from the nullity and/or rank of T?

T is not one-to-one. We can tell because the nullity of T is not zero.

e. Is T onto? How can you tell from the nullity and/or rank of T?

T is not onto. We can tell because the rank of T does not equal the dimension of the codomain.