Math 24
Spring 2012
Problems from Monday April 9

First some definitions. If W_1 and W_2 are two subspaces of V, we define

$$W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1 \;\&\; w_2 \in W_2\}.$$

In other words, $W_1 + W_2$ is the collection of all vectors you can get by adding an element of W_1 to an element of W_2. If $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$, then we say V is the *direct sum* of W_1 and W_2, and we write $V = W_1 \oplus W_2$.

1. Prove that $W_1 + W_2$ is the smallest subspace containing both W_1 and W_2. (In other words, $W_1 + W_2$ is the span of $W_1 \cup W_2$.)

2. Give examples of pairs of subspaces W_1 and W_2 of $\mathbb{R}^3$, neither of which is contained in the other, such that:

   (a) $W_1 + W_2 \neq \mathbb{R}^3$. In your example, what is $W_1 + W_2$?

   (b) $W_1 + W_2 = \mathbb{R}^3$, but $\mathbb{R}^3$ is not the *direct* sum of W_1 and W_2. In your example, what is $W_1 \cap W_2$?

   (c) $\mathbb{R}^3$ is the direct sum of W_1 and W_2.

3. Suppose W_1 and W_2 are both subspaces of a finite-dimensional vector space V. Make a conjecture about the relationship among the dimensions of W_1, W_2, $W_1 \cap W_2$, and $W_1 + W_2$.

4. Express $M_{2\times 2}(\mathbb{C})$ as the direct sum of two nonzero subspaces.

5. Express $P(\mathbb{R})$ as the direct sum of two nonzero subspaces in two ways.

   (a) One of the subspaces has finite dimension.

   (b) Both of the subspaces are infinite-dimensional.

6. Prove the conjecture you made in problem (3). Hint: A basis $\{x_1, \ldots, x_k\}$ for $W_1 \cap W_2$ can be extended to a basis $\{x_1, \ldots, x_k, y_1, \ldots y_n\}$ for W_1. It can also be extended to a basis $\{x_1, \ldots, x_k, z_1, \ldots z_m\}$ for W_2. For homework, you might want to verify your conjecture by looking at problem 29(a) of section 1.6 of the textbook. Please make a conjecture yourself first, though.

7. Every vector in $W_1 + W_2$ can be expressed as a sum, $w_1 + w_2$, of vectors $w_1 \in W_1$ and $w_2 \in W_2$. In what cases is this expression *unique*? Prove your answer is correct.