Math 24
Spring 2012
Questions (mostly) from the Textbook

1. TRUE OR FALSE ?

   (a) The zero vector space has no basis. (F)

   (b) Every vector space that is generated by a finite set has a basis. (T)

   (c) Every vector space has a finite basis. (F)

   (d) A vector space cannot have more than one basis. (F)

   (e) If a vector space has a finite basis, then the number of vectors in every basis is the same. (T)

   (f) The dimension of $P_n(F)$ is n. (F)

   (g) The dimension of $M_{m\times n}(F)$ is $m+n$. (F)

   (h) Suppose that V is a finite-dimensional vector space, that S_1 is a linearly independent subset of V, and that S_2 is a subset of V that generates V. Then S_1 cannot contain more vectors than S_2. (T)

   (i) If S generates the vector space V, then every vector in V can be written as a linear combination of vectors in S in only one way. (F)

   (j) Every subspace of a finite-dimensional vector space is finite-dimensional. (T)

   (k) If V is a vector space having dimension n, then V has exactly one subspace with dimension 0 and exactly one subspace with dimension n. (T)

   (l) If V is a vector space having dimension n, and S is a subset of V with n vectors, then S is linearly independent if and only if S spans V. (T)

   (m) If a vector space has an infinite basis, then every basis is infinite. (T)

   (n) If V has an infinite basis, and S is an infinite linearly independent subset of V, then S must be a basis for V. (F)

2. Determine whether the set $\{(1,2,1),(1,4,-1),(3,8,2)\} \subseteq \mathbb{R}^3$ is linearly independent.

   We can always do this by solving

   $$a(1,2,1)+b(1,4,-1)+c(3,8,2)=(0,0,0);$$

   the set is linearly independent just in case $(a,b,c)=(0,0,0)$ is the only solution. In this example, the set is linearly independent.

3. Determine whether the set $\{1+x, x+x^2, x^2+x^3, x^3+1\} \subseteq P_3(\mathbb{Q})$ is linearly independent.

   A slightly different way to do this is to ask whether any of these polynomials can be written as a linear combination of the earlier ones. (This works because of the result we proved in problem (5) on Wednesday.) The only possibility is the fourth one (since the others are all of higher degree than any of the preceding ones). So we need to see whether we can write

   $$x^3+1=a(x^3+x^2)+b(x^2+x)+c(x+1).$$

   In order to get x^3 on the left, we must have $a=1$; then, in order to have no x^2 term on the left, we must have $b=-1$; then in order to have no x term on the left, we must have $c=1$. We can check that $(a,b,c)=(1,-1,1)$ works. Therefore the set is not linearly independent.

4. Without using the results of Section 1.6, give a proof by induction on n of the following proposition:

   If V is a vector space, and $S = \{x_1, x_2, \ldots, x_n\}$ generates V, then there is a linearly independent subset of S that also generates V.

   (Hint: If S is not already linearly independent, you can express one of the vectors in S as a linear combination of the others.)

   As the base case, you may consider $n = 0$ or $n = 1$. In the case $n = 0$, since S has no elements, $S = \emptyset$, and S is already linearly independent. Therefore S is a linearly independent subset of S that generates V.

   In the case $n = 1$, say $S = \{x_1\}$. If $x_1 \neq 0$, then S is linearly independent, and we are done. If $x_1 = 0$, then S generates the zero vector space, and $\emptyset$ is a linearly independent subset of S that also generates the zero vector space.

   For the inductive step, assume as inductive hypothesis that if a set of n vectors generates V, then it has a linearly independent subset that generates V. Now let S be a set of $n + 1$ vectors that generates V.

   If S is linearly independent, then it itself is a linearly independent subset that generates V.

   If not, there is a vector $v \in S$ that is a linear combination of other vectors in S. That is, we can write $S = S' \cup \{v\}$, where $v \in span(S')$. This implies that $span(S') = span(S' \cup \{v\}) = span(S) = V$. Because S' is a set of n vectors, by inductive hypothesis, S' has a linearly independent subset that spans V.

5. Let V be a vector space over a field F of characteristic not equal to two. (Remember that the characteristic of F is two just in case $1 + 1 = 0$, as in $\mathbb{Z}_2$, so we are assuming that in F we have $1 + 1 \neq 0$. If this is too confusing to start with, first do part (a) in the case the field of scalars F is equal to $\mathbb{R}$. Then try to convince yourself that the only special thing about $\mathbb{R}$ you used is that $1 + 1 \neq 0$.)

   (a.) Let u and v be two distinct nonzero vectors in V. Prove that $\{u, v\}$ is linearly independent if and only if $\{u + v,\ u - v\}$ is linearly independent.

   Suppose that $\{u, v\}$ is linearly independent. We must show that $\{u + v, u - v\}$ is also linearly independent. Suppose some linear combination equals zero,

$$a(u + v) + b(u - v) = 0.$$

   We must show that $a = 0$ and $b = 0$. To do this, rewrite the equation as

$$(a + b)u + (a - b)v = 0.$$

   Because $\{u, v\}$ is linearly independent, we know $a + b = 0$ and $a - b = 0$. Adding these two equations gives $a + a = 0$; adding -1 times the second to the first gives $b + b = 0$.

   If we are working over $\mathbb{R}$ we can immediately conclude $a = 0$ and $b = 0$. If not, then we can rewrite the equations as $a(1 + 1) = 0$ and $b(1 + 1) = 0$. Since we assume that F does NOT have characteristic two, $1 + 1 \neq 0$, and we can multiply through by $(1 + 1)^{-1}$ to get $a = 0$ and $b = 0$.

   This is only one direction, of course. Now you have to do the converse.

   (b.) What happens if F has characteristic equal to two?

   In that case, $(u + v) + (u - v) = u + u = (1 + 1)u = 0u = 0$. This is a nontrivial linear combination equal to zero, so $\{u + v, u - v\}$ appears to be linearly dependent.

   But not so fast! In fact, since $1 + 1 = 0$, we have $1 = -1$, so $v = -v$, and $u + v = u - v$. Really, then, $\{u + v, u - v\}$ contains only one vector, which can be expressed as either $u+v$ or $u-v$. Since $u \neq v$, the vector $u-v$ is nonzero, so regardless of whether $\{u, v\}$ is linearly independent, $\{u+v, u-v\}$ is a linearly independent set containing a single vector.