Math 24
Spring 2012
Friday, March 30

1. In the last class we said that the subspaces of $\mathbb{R}^2$ are

   (a) The zero subspace, $\{(0, 0)\}$.

   (b) Any line through the origin.

   (c) The entire space $\mathbb{R}^2$.

   What are the subspaces of $\mathbb{R}^3$?

   (You do not have to prove your answer is correct.)

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   There are four possibilities now:

   (a) The zero subspace, $\{(0, 0)\}$.

   (b) Any line through the origin.

   (c) Any plane through the origin.

   (d) The entire space $\mathbb{R}^3$.

   We will see later that this is a general fact: An n-dimensional vector space has subspaces of every dimension between 0 and n.

2. What is the smallest subspace of $\mathbb{R}^3$ containing the vectors $(1,0,0)$ and $(0,1,1)$. (Give an algebraic description, not a geometric one.)

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It is the subspace W consisting of all vectors of the form (a,b,b). (Or, all vectors (x,y,z) satisfying the equation $y-z=0$.)

First, we can see that W is a subspace, because it contains $(0,0,0)$, the sum of two vectors with equal y- and z-components has equal y- and z-components, and a scalar multiple of a vector with equal y- and z-components has equal y- and z-components. (W is closed under addition and multiplication by scalars.)

Clearly W contains $(1,0,0)$ and $(0,1,1)$.

To see that W is the smallest possible such subspace, show that any subspace W' containing $(1,0,0)$ and $(0,1,1)$ must contain every vector in W. This is true because every vector $\vec{w} \in W$ can be written in the form

$$\vec{w} = (a,b,b) = a(1,0,0) + b(0,1,1),$$

and since $\vec{W}'$ contains $(1,0,0)$ and $(0,1,1)$ and is closed under multiplication by scalars and under addition, W' must also contain $a(1,0,0) + b(0,1,1)$.

3. What is the smallest subspace of $P_2(\mathbb{R})$ containing the polynomials x^2 and $x+1$?

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By exactly the same reasoning, it consists of all polynomials of the form $ax^2 + bx + b$.

4. Suppose V is a vector space and $\vec{x}$ and $\vec{y}$ are elements of V. What can you say about the smallest subspace of V containing $\vec{x}$ and $\vec{y}$?

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It consists of all vectors of the form $a\vec{x} + b\vec{y}$.

Similarly, the smallest subspace of V containing $\vec{x}$, $\vec{y}$, and $\vec{z}$ consists of all vectors of the form $a\vec{x} + b\vec{y} + c\vec{z}$.

Some vocabulary that the textbook will define shortly:

A vector of the form $a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_n\vec{x}_n$ is called a *linear combination* of $\vec{x}_1$, $\vec{x}_2$, $\ldots$, $\vec{x}_n$. The collection of all linear combinations of vectors from a set of vectors X is called the *span* of X.

It makes sense that the span of X is the smallest subspace containing X. It is also called the subspace *generated* by X.

5. Show that , for any real numbers a, b, c, and d, the set of matrices $\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ in $M_{2\times 2}(\mathbb{R})$ whose entries satisfy the equation

$$a(m_{11}) + b(m_{12}) + c(m_{21}) + d(m_{22}) = 0$$

is a subspace of $M_{2\times 2}(\mathbb{R})$.

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To show $\vec{W}$ is a subspace, we must show that W contains the zero vector, in this case $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and is closed under addition and multiplication by scalars.

It is clear that the entries of the zero matrix satisfy this equation.

For closure under addition, suppose the entries of $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ and $N = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}$ satisfy the equation. Then the entries of

$$M + N = \begin{pmatrix} m_{11} + n_{11} & m_{12} + n_{12} \\ m_{21} + n_{21} & m_{22} + n_{22} \end{pmatrix}$$

also satisfy the equation, since

$$a(m_{11} + n_{11}) + b(m_{12} + n_{12}) + c(m_{21} + n_{21}) + d(m_{22} + n_{22}) =$$

$$a(m_{11}) + b(m_{12}) + c(m_{21}) + d(m_{22}+) + a(n_{11}) + b(n_{12}) + c(n_{21}) + d(n_{22}) =$$

$$0 + 0 = 0.$$

Closure under multiplication by scalars is similar.

6. Show that the set of all differentiable functions f from $\mathbb{R}$ to $\mathbb{R}$ that satisfy the differential equation

$$\frac{d^2 f}{dx^2} = -f$$

is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ (the space of all functions from $\mathbb{R}$ to $\mathbb{R}$).

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We need to show that the zero function satisfies this equation (which is true, since $\frac{d^2\, 0}{dx^2} = 0$), and that if f and g satisfy the equation,

$$\frac{d^2 f}{dx^2} = -f \qquad \& \qquad \frac{d^2 g}{dx^2} = -g,$$

then so do $f + g$ and cf. This is true because

$$\frac{d^2(f+g)}{dx^2} = \frac{d^2 f}{dx^2} + \frac{d^2 g}{dx^2} = (-f) + (-g) = -(f+g)$$

and

$$\frac{d^2(cf)}{dx^2} = c\frac{d^2 f}{dx^2} = c(-f) = -(cf).$$

This turns out to be the subspace generated by the functions $\sin x$ and $\cos x$, all functions of the form $a \sin x + b \cos x$. Because this subspace is generated by two functions, and cannot be generated by a single function, it is said to be two-dimensional.

7. Let W be the smallest subspace of $P_3(\mathbb{R})$ containing the polynomials x^3+x^2, x^2+x, and $x+1$. Determine whether the polynomial x^3-x is in W.

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W consists of all polynomials of the form $s(x^3+x^2)+t(x^2+x)+u(x+1)$, so we must see whether x^3-x can be written in this form. That is, we need to see whether we can find numbers s, t and u satisfying

$$s(x^3+x^2)+t(x^2+x)+u(x+1)=x^3-x,$$

that is,

$$sx^3+(s+t)x^2+(t+u)x+u=x^3-x=1(x^3)+0(x^2)+(-1)(x)+0.$$

Because equal polynomials have the same coefficients, that means we need to see whether we can solve

$$s=1$$

$$s+t=0$$

$$t+u=-1$$

$$u=0$$

This system is not too hard to solve; we get $s=1$, $t=-1$, $u=0$ for a solution. Therefore $x^3-x=(x^3+x^2)-(x^2+x)$, and x^3-x is in fact in W.

This example leads us into the question of solving systems of linear equations, which comes up next in the reading.