Math 24
Spring 2012
Wednesday, May 23

(1.) TRUE or FALSE? (These questions always deal with finite-dimensional inner product spaces.)

(a.) Every self-adjoint operator is normal.

(b.) Operators and their adjoints have the same eigenvectors.

(c.) If T is an operator on an inner product space V, then T is normal if and only if $[T]_\beta$ is normal, where β is any ordered basis for V.

(d.) A real or complex matrix A is normal if and only if L_A is normal.

(e.) The eigenvalues of a self-adjoint operator must all be real.

(f.) The identity and zero operators are self-adjoint.

(g.) Every normal operator is diagonalizable.

(h.) Every self-adjoint operator is diagonalizable.

Solutions are in the back of the textbook.

(2.) Determine whether the linear operator is normal, self-adjoint, or neither. If possible, give an orthonormal basis of eigenvectors, with corresponding eigenvalues.

(a.) $V = \mathbb{R}^2$, and $T(a,b) = (2a - 2b,\ -2a + 5b)$.

The matrix of A in the standard basis is $\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$, which is self-adjoint. so T is self-adjoint.

The characteristic polynomial of A is $(2-\lambda)(5-\lambda) - 4 = \lambda^2 - 7\lambda + 6 = (\lambda - 6)(\lambda - 1)$, so the eigenvalues are $\lambda = 1$ and $\lambda = 6$.

An eigenvector corresponding to $\lambda = 1$ is $(2, 1)$, and an eigenvector corresponding to $\lambda = 6$ is $(1, -2)$. These vectors are orthogonal, so $\beta = \{\frac{\sqrt{5}}{5}(2,1),\ \frac{\sqrt{5}}{5}(1,-2)\}$ is an orthonormal basis of eigenvectors.

(b.) $V = \mathbb{C}^2$, and $T(a,b) = (2a + ib,\ a + 2b)$.

The matrix of A in the standard basis is $\begin{pmatrix} 2 & i \\ 1 & 2 \end{pmatrix}$, which is not self-adjoint. so T is not self-adjoint.

(c.) $V = M_{2\times 2}(\mathbb{R})$ and $T(A) = A^t$.

The matrix of A in the standard basis is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, which is self-adjoint. so T is self-adjoint.

The characteristic polynomial of A is $(1-\lambda)^2(\lambda^2 - 1) = (1-\lambda)^2(1-\lambda)(1+\lambda)$, so the eigenvalues are $\lambda = 1$, with multiplicity 3, and $\lambda = -1$, with multiplicity 1.

Three eigenvectors corresponding to $\lambda = 1$ are $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and an eigenvector corresponding to $\lambda = -1$ is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. These vectors are orthogonal, so $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$ is an orthonormal basis of eigenvectors.

(3.) Is $\{A \in M_{n\times n}(\mathbb{R}) \mid A \text{ is self-adjoint}\}$ a subspace of $M_{2\times 2}(\mathbb{R})$?

Yes. It is not hard to show that 0 is self-adjoint, that if A is self-adjoint so is cA (since $(cA)^t = c(A^t)$, and over $\mathbb{R}$, the adjoint of a matrix is its transpose), and that if A and B are self-adjoint, so is $A + B$ (because $(A+B)^t = A^t + B^t$).

Is $\{A \in M_{n\times n}(\mathbb{C}) \mid A \text{ is self-adjoint}\}$ a subspace of $M_{2\times 2}(\mathbb{C})$?

No. This set is not closed under multiplication by scalars: the identity matrix I is self-adjoint, but iI is not, because $(iI)^* = (-i)I \neq iI$.

Cultural Enrichment

Here is an important example of an infinite-dimensional inner product space, with significance in mathematics and in theoretical physics. It is called Hilbert space.

Define an infinite sequence of real numbers, $(a_1, a_2, a_3, \ldots, a_n \ldots)$ to be *square summable* if $\sum_{i=1}^{\infty}(a_i)^2$ converges. Let $\mathbb{H}$ be the set of all square summable sequences. Define addition and scalar multiplication on $\mathbb{H}$ coordinatewise:

$$(a_1, a_2, a_3, \ldots, a_n \ldots) + (b_1, b_2, b_3, \ldots, b_n \ldots) = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3,\ \ldots,\ a_n + b_n \ldots)$$

$$c(a_1, a_2, a_3, \ldots, a_n \ldots) = (ca_1, ca_2, ca_3, \ldots, ca_n \ldots).$$

(1.) Verify that $\mathbb{H}$ is an infinite-dimensional vector space.

You will need to show that the sum of two square-summable sequences is itself square-summable. If you have forgotten everything you ever knew about infinite series, you can take this for granted (for now).

To show $\mathbb{H}$ is a vector space, first we can note that $\mathbb{H}$ is a subset of the vector space consisting of all infinite sequences of real numbers, so we need to check that $\mathbb{H}$ is closed under multiplication by scalars and under addition.

For multiplication by scalars, suppose $(a_1, a_2, a_3, \ldots) \in \mathbb{H}$, so $\sum_{i=1}^{\infty}(a_i)^2$ converges, say $\sum_{i=1}^{\infty}(a_i)^2 = a$. Then $\sum_{i=1}^{\infty}(ca_i)^2 = \sum_{i=1}^{\infty} c^2(a_i)^2 = c^2\sum_{i=1}^{\infty}(a_i)^2 = c^2 a$, and so $c(a_1, a_2, a_3, \ldots) \in \mathbb{H}$.

For addition, suppose $(a_1, a_2, a_3, \ldots) \in \mathbb{H}$ and $(b_1, b_2, b_3, \ldots) \in \mathbb{H}$, so $\sum_{i=1}^{\infty}(a_i)^2 = a$ and $\sum_{i=1}^{\infty}(b_i)^2 = b$. Then $\sum_{i=1}^{\infty}(a_i + b_i)^2 = \sum_{i=1}^{\infty}((a_i)^2 + 2a_ib_i + (b_i)^2)$. We can see that either $|a_ib_i| \leq a_i^2$ or $|a_ib_i \leq b_i^2|$, and so $(a_i)^2 + 2a_ib_i + (b_i)^2) \leq 3a_i^2 + 3b_i^2$. Since $\sum_{i=1}^{\infty} 3a_i^2 + 3b_i^2$ converges (to $3a + 3b$), by the comparison test, so does $\sum_{i=1}^{\infty}((a_i)^2 + 2a_ib_i + (b_i)^2)$.

To show $\mathbb{H}$ is infinite-dimensional, show there is an infinite, linearly independent set of vectors. The the vectors e_n, where the i^{th} coordinate of e_n is 1 if $i = n$ and 0 if $i \neq n$, form such a set.

We define an inner product on $\mathbb{H}$ by

$$\langle (a_1, a_2, a_3, \ldots, a_n \ldots), (b_1, b_2, b_3, \ldots, b_n \ldots) \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n + \cdots .$$

(2.) Show that this is in fact an inner product. The first thing you need to show is that this sum actually converges.

To show the sum converges, we can again use the comparison test, since $|a_i b_i| \leq a_i^2 + b_i^2$. It is not too hard to check the four properties of an inner product.

(3.) Show that the vectors e_n, where the i^{th} coordinate of e_n is 1 if $i = n$ and 0 if $i \neq n$, form an orthonormal set.

Compute $\langle e_i, e_j \rangle = \delta_{ij}$.

(4.) Show that the orthonormal set $\{e_n \mid n \in \mathbb{N}\}$ is not a basis and cannot be extended to an orthonormal basis.

This set does not span $\mathbb{H}$, because any linear combination of the e_n has only finitely many non-zero entries. Therefore $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right)$ is an example of a vector in $\mathbb{H}$ not in the span.

On the other hand, if $a = (a_1, a_2, a_3, \ldots)$ is orthogonal to every e_n, then for every n we must have $0 = \langle a, e_n \rangle = a_n$. This can only happen if $a = (0, 0, 0, \ldots)$. Therefore, since only the zero vector is orthogonal to every e_n, there is no way to extend the set of e_n to an orthonormal basis.

(5.) Convince yourself that every vector in $\mathbb{H}$ is the limit of a sequence of vectors in $span(\{e_1, e_2, e_3, \ldots, e_n \ldots\})$, where the notion of limit is defined using the notion of distance that comes from this inner product (the distance between two vectors is the norm of their difference). When thinking about the span, remember that even though $\mathbb{H}$ has infinite dimension, we can only take finite linear combinations of vectors.

Show that $a = (a_1, a_2, a_3, \ldots)$ is the limit of the sequence of vectors $(a_1, 0, 0, 0 \ldots)$, $(a_1, a_2, 0, 0 \ldots)$, $(a_1, a_2, a_3, 0 \ldots)$, $\ldots$