Math 24
Spring 2012
Friday, May 18
Sample Solutions

(1.) TRUE or FALSE?

(a.) The Gram-Schmidt orthogonalization process allows us to construct an orthonormal set from an arbitrary set of vectors.
(b.) Every nonzero finite-dimensional inner product space has an orthonormal basis.
(c.) The orthogonal complement of any set is a subspace.
(d.) If $\{v_1, v_2, \ldots, v_n\}$ is a basis for an inner product space V, then for any $x \in V$, the scalars $\langle x, v_i \rangle$ are the Fourier coefficients of x.
(e.) An orthonormal basis must be an ordered basis.
(f.) Every orthogonal set is linearly independent.
(g.) Every orthonormal set is linearly independent.

Answers are in the back of the book.

(2.) Let $v_1 = (1, 1, 0)$, $v_2 = (1, 0, 1)$, and $v_3 = (0, 1, 1)$ in $\mathbb{R}^3$. Use the Gram-Schmidt orthogonalization process to find an orthonormal basis $\{w_1, w_2, w_3\}$ for $\mathbb{R}^3$ such that $span(\{w_1\}) = span(\{v_1\})$ and $span(\{w_1, w_2\}) = span(\{v_1, v_2\})$.

$x_1 = v_1 = (1, 1, 0)$

$x_2 = v_2 - (\ x_1\text{-component}(v_2)) = (1, 0, 1) - \frac{\langle (1,0,1), (1,1,0) \rangle}{\langle (1,1,0), (1,1,0) \rangle}(1, 1, 0) = (1, 0, 1) - \frac{1}{2}(1, 1, 0) = (\frac{1}{2}, -\frac{1}{2}, 1)$.

$x_3 = v_3 - (\ x_1\text{-component}(v_3)) - (\ x_2\text{-component}(v_3)) = (0, 1, 1) - \frac{\langle (0,1,1), (1,1,0) \rangle}{\langle (1,1,0), (1,1,0) \rangle}(1, 1, 0) - \frac{\left\langle (0,1,1), (\frac{1}{2}, -\frac{1}{2}, 1) \right\rangle}{\left\langle (\frac{1}{2}, -\frac{1}{2}, 1), (\frac{1}{2}, -\frac{1}{2}, 1) \right\rangle}(\frac{1}{2}, -\frac{1}{2}, 1) = (0, 1, 1) - \frac{1}{2}(1, 1, 0) - \frac{1}{3}(\frac{1}{2}, -\frac{1}{2}, 1) = (-\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$

$w_1 = \frac{1}{||x_1||} x_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$
$w_2 = \frac{1}{||x_2||} x_2 = (\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3})$
$w_3 = \frac{1}{||x_3||} x_3 = (-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$

(3.) Find a basis for the orthogonal complement of W. (Problem (2) should be useful here.)

We use what we know about dimension.

(a.) $W = \{s(1,1,0) + t(1,0,1) \mid s,t \in \mathbb{R}\}$

$W = span(v_1, v_2) = span(w_1, w_2)$. We need one vector orthogonal to W to span $W^\perp$, and w_3 will do. A basis is $\{(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})\}$.

(b.) $W = \{s(1,0,1) \mid s \in \mathbb{R}\}$

Here W is $span(v_2)$, a subspace of the space in (a), so w_3 is orthogonal to W, and we need another vector orthogonal to W to form a basis. In general, we could use Gram-Schmidt orthogonalization, and take

$v_1 - (v_2\text{-component}(v_1)) = (1,1,0) - \dfrac{\langle(1,1,0),(1,0,1)\rangle}{\langle(1,0,1),(1,0,1)\rangle}(1,0,1) = (1,1,0) - \frac{1}{2}(1,0,1) = (\frac{1}{2}, 1, -\frac{1}{2})$. A basis is $\{(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}), (\frac{1}{2}, 1, -\frac{1}{2})\}$.

A nice thing about this basis is that it is orthogonal.

Complete these problems from Wednesday's worksheet (if you didn't already):

(5.) Suppose that $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V.

(a.) Given $x = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ and $y = b_1v_1 + b_2v_2 + \cdots + b_nv_n$, find a formula for $\langle x, y\rangle$ in terms of the a_i, the b_i, and the inner products $\langle v_i,\, v_j\rangle$.

(b.) Show that if β is an orthonormal set, then $\langle x, y\rangle = \langle [x]_\beta,\, [y]_\beta\rangle$, where $\langle [x]_\beta,\, [y]_\beta\rangle$ is the standard inner product in F^n. (A basis for V that is an orthonormal set is called an orthonormal basis.)

(6.) Suppose that β is any basis for V, possibly infinite. Define an inner product on V as follows:

Given any x and y in V, we can find $v_1, v_2, \ldots, v_n$ in β such that $x = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ and $y = b_1v_1 + b_2v_2 + \cdots + b_nv_n$. Set $\langle x, y\rangle = a_1\bar{b}_1 + a_2\bar{b}_2 + \cdots + a_n\bar{b}_n$.

(a.) Show that this is an inner product.

(b.) Show that β is an orthonormal set for this inner product.

The solutions are in the solutions to Wednesday's problems.

(7.) Prove that if V is an n-dimensional inner product space, and W is an m-dimensional subspace of V, then:

(a.) $W = (W^{\perp})^{\perp}$

(b.) V is the direct sum of W and $W^{\perp}$

(c.) There is an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ for V such that $\{v_1, v_2, \ldots, v_m\}$ is an orthonormal basis for W.

(a): First, suppose $w \in W$. Then (by definition of $W^{\perp}$) w is orthogonal to everything in $W^{\perp}$, so $w \in (W^{\perp})^{\perp}$. This shows $W \subseteq (W^{\perp})^{\perp}$.

Now, suppose that $w \in (W^{\perp})^{\perp}$; we must show $w \in W$. By Theorem 6.6, we can find vectors $u \in W$ and $v \in W^{\perp}$ such that $w = u + v$. Because $v \in W^{\perp}$ and $u \in W$ we have $\langle u, v \rangle = 0$. Now

$$\langle w, v \rangle = \langle u + v, v \rangle = \langle u, v \rangle + \langle v, v \rangle = \langle v, v \rangle$$

Because $v \in W^{\perp}$, and $w \in (W^{\perp})^{\perp}$, we have that $\langle w, v \rangle = 0$, thus from the above we have $\langle v, v \rangle = 0$, so $v = 0$.

(b): This follows directly from Theorem 6.6. Every $v \in V$ can be written as $v = u + w$ for $u \in W$ and $v \in W^{\perp}$, so $W + W^{\perp} = V$. Also $W \cap W^{\perp} = \{0\}$, because if $v \in W$ and $v \in W^{\perp}$, because every element of W is orthogonal to every element in $W^{\perp}$, we have $\langle v, v \rangle = 0$, so $v = 0$.

In fact, (b) is practically a restatement of Theorem 6.6.

(c): Let $\beta = \{v_1, v_2, \ldots, v_m\}$ be any orthonormal basis for W. (Every finite-dimensional inner product space has an orthonormal basis by Theorem 6.5.) By Theorem 6.7, we can extend β to an orthonormal basis for V.

One more problem in case you got through the others. Note, a good way to get started thinking about problems involving V/W is to think about a concrete example, like the one in which $V = \mathbb{R}^2$, W is the x-axis, and V/W is the set of horizontal lines.

(8.) Suppose that W is a subspace of a finite-dimensional inner product space V. Let us try to define an inner product on V/W by

$$\langle x + W,\, y + W \rangle = \langle x, y \rangle \,.$$

Is this function well-defined? If so, does it actually give an inner product on V/W?

If the answer to either of those questions is "no," can you think of a way you could use the inner product on V to define an inner product on V/W?

This function is not well-defined.

For it to be well-defined, it would have to be the case that if $x + W = x' + W$ and $y + W = y' + W$, then $\langle x, y \rangle = \langle x', y' \rangle$. But this is not generally true.

For example, suppose $V = \mathbb{R}^2$ and W is the x-axis. Then $W = (0,0) + W = (1,0) + W$, but if we take $x = y = (0,0)$ and $x' = y' = (1,0)$, then $\langle x, y \rangle = \langle (0,0), (0,0) \rangle = 0$ and $\langle x', y' \rangle = \langle (1,0), (1,0) \rangle = 1$.

A hint on the last question: In this example, we can reasonably define $\langle (a,b) + W, (c,d) + W \rangle = \langle (0,b), (0,d) \rangle$. Roughly speaking, nonzero vectors in W create problems, so we eliminate the "W-component" before taking the inner product.