Math 24
Spring 2012
Wednesday, May 16
Sample Solutions

(1.) TRUE or FALSE?

(a.) An inner product is a scalar-valued function on the set of ordered pairs of vectors.
(b.) An inner product space must be over the field of real or complex numbers.
(c.) An inner product is linear in both components.
(d.) There is exactly one inner product on the vector space $\mathbb{R}^n$.
(e.) The triangle inequality holds only in finite-dimensional inner product spaces.
(f.) Only square matrices have a conjugate-transpose.
(g.) If x, y, and z are vectors in an inner product space such that $\langle x, y\rangle = \langle x, z\rangle$ then $y = z$.
(h.) If $\langle x, y\rangle = 0$ for all x in an inner product space, then $y = 0$.

Answers are in the back of the textbook.

(2.) Provide reasons why each of the following is not an inner product on the given vector space.

In each case we give an example of $x \neq 0$ with $\langle x, x\rangle \not> 0$.

(a.) $\langle (a, b),\, (c, d)\rangle = ac + bd$ on $\mathbb{C}^2$.

$$\langle (0, i), (0, i)\rangle = -1$$

(b.) $\langle (a, b),\, (c, d)\rangle = ac - bd$ on $\mathbb{R}^2$.

$$\langle (0, 1), (0, 1)\rangle = -1$$

(c.) $\langle A, B\rangle = tr(A + B)$ on $M_{2\times 2}(\mathbb{R})$.

$$\left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle = 0$$

(d.) $\langle f(x),\, g(x)\rangle = \int_0^1 f'(t)g(t)\, dt$ on $P(\mathbb{R})$ where $'$ denotes differentiation.

$$\langle 1, 1\rangle = 0$$

(3.)

(a.) Is $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ an orthonormal set in $M_{2\times 2}(\mathbb{C})$ with the Frobenius inner product $\langle A, B\rangle = tr(B^*A)$?

Yes.

(b.) Define an inner product on $M_{2\times 2}(\mathbb{C})$ that is different from the Frobenius inner product.

The easy way is, for example, $\langle A, B\rangle = 2(tr(B^*A))$. For a different example, set $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + bx + cx^2 + dx^3$, and then set $\langle A, B\rangle = \int_0^1 T(A)T(B)dx$.

(This uses the fact that $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$ is an inner product on $P(\mathbb{R})$. We let T be a one-to-one linear transformation to another inner product space, and "pull back" that other inner product by T.)

(4.) Suppose that $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V.

(a.) Given $x = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ and $y = b_1v_1 + b_2v_2 + \cdots + b_nv_n$, find a formula for $\langle x, y\rangle$ in terms of the a_i, the b_i, and the inner products $\langle v_i,\, v_j\rangle$.

We use the fact that the inner product is linear in the first component, so

$$\langle a_1v_1 + a_2v_2 + \cdots + a_nv_n, y\rangle = a_1\langle v_1, y\rangle + a_2\langle v_2, y\rangle + \cdots + a_n\langle v_n, y\rangle\,,$$

and conjugate linear in the second component, so

$$\langle v_i, b_1v_1 + b_2v_2 + \cdots + b_nv_n\rangle = \overline{b_1}\langle v_i, v_1\rangle + \overline{b_2}\langle v_i, v_2\rangle + \cdots + \overline{b_n}\langle v_i, v_n\rangle\,,$$

to get

$$\langle x, y\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i\overline{b_j}\langle v_i, v_j\rangle\,.$$

(b.) Show that if β is an orthonormal set, then $\langle x, y\rangle = \langle [x]_\beta,\, [y]_\beta\rangle$, where $\langle [x]_\beta,\, [y]_\beta\rangle$ is the standard inner product in F^n. (A basis for V that is an orthonormal set is called an orthonormal basis.)

If β is an orthonormal basis, then $\langle v_i, v_j\rangle$ is 1 if $i = j$, and 0 otherwise, so we have

$$\langle x, y\rangle = \sum_{i=1}^{n} a_i\overline{b_i} = \langle (a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)\rangle = \langle [x]_\beta,\, [y]_\beta\rangle\,.$$

(5.) Suppose that β is any basis for V, possibly infinite. Define an inner product on V as follows:

Given any x and y in V, we can find $v_1, v_2, \ldots, v_n$ in β such that $x = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ and $y = b_1v_1 + b_2v_2 + \cdots + b_nv_n$. Set $\langle x, y\rangle = a_1\overline{b}_1 + a_2\overline{b}_2 + \cdots + a_n\overline{b}_n$.

(a.) Show that this is an inner product.

(b.) Show that β is an orthonormal set for this inner product.

The homework assignment from today includes the special case of this problem when V has finite dimension, so I will not post the solution to this one.

Notice that this problem, together with the previous problem and the fact that a finite-dimensional inner product space always has an orthonormal basis, says that inner products, like other properties of finite-dimensional vector spaces, can be viewed at the coordinate level:

If β is an orthonormal basis for an inner product on V, then we have $\langle x, y\rangle = \langle [x]_\beta, [y]_\beta\rangle$, by problem (4).

On the other hand, by the homework problem assigned today, if β is any ordered basis, then we can define an inner product on V by $\langle x, y\rangle = \langle [x]_\beta, [y]_\beta\rangle$; for this inner product, β is an orthonormal basis.