Math 24
Spring 2012
Friday, May 11
Sample Solutions

1. TRUE or FALSE?

   (a) Every linear operator on an n-dimensional space has n distinct eigenvalues.
   (b) If a real matrix has one eigenvector, then it has an infinite number of eigenvectors.
   (c) There exists a square matrix with no eigenvectors.
   (d) Eigenvalues must be nonzero scalars.
   (e) Any two eigenvectors are linearly independent.
   (f) The sum of two eigenvalues of a linear operator T is also an eigenvalue of T.
   (g) Linear operators on infinite-dimensional vector spaces never have eigenvalues.
   (h) An $n \times n$ matrix A with entries from a field F is similar to a diagonal matrix if and only if there is a basis fro F^n consisting of eigenvalues of A.
   (i) Similar matrices always have the same eigenvalues.
   (j) Similar matrices always have the same eigenvectors.
   (k) The sum of two eigenvectors of T is always an eigenvector of T.

   Answers are in the back of the book.

2. Find the eigenvalues, and a basis for each of the corresponding eigenspaces, of the matrix $\begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$. Is there a basis for $\mathbb{R}^3$ consisting of eigenvectors of this matrix?

   The eigenvalues of A are the roots λ of the characteristic polynomial, $det(A - \lambda I)$. In this case the characteristic polynomial is $-(1-\lambda)^2(1+\lambda)$, so the eigenvalues are $\lambda = 1$ (of multiplicity 2) and $\lambda = -1$ (of multiplicity 1).

   The eigenspace corresponding to λ is the null space of $A - \lambda I$, which we know how to find by row reducing the matrix $A - \lambda I$. For $\lambda = 1$ a basis for the eigenspace is $\{(1, 1, 0), (1, 0, -1)\}$, and for $\lambda = -1$ a basis for the eigenspace is $\{(1, 0, 1)\}$.

   There is a basis of eigenvectors for $\mathbb{R}^3$, namely $\{(1, 1, 0), (1, 0, -1), (1, 0, 1)\}$.

3. Find the eigenvalues, and a basis for each of the corresponding eigenspaces, of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$. Is there a basis for $\mathbb{R}^3$ consisting of eigenvectors of this matrix?

   Once again, the characteristic polynomial is $-(1-\lambda)^2(1+\lambda)$, so the eigenvalues are $\lambda = 1$ (of multiplicity 2) and $\lambda = -1$ (of multiplicity 1).

   For $\lambda = 1$ a basis for the eigenspace is $\{(1,0,0)\}$, and for $\lambda = -1$ a basis for the eigenspace is $\{(1,2,-4)\}$.

   Since 1 is an eigenvalue of multiplicity 2, but the corresponding eigenspace has dimension 1, there is not a basis of eigenvectors for $\mathbb{R}^3$.

4. For each of the matrices A in problems (2) and (3) that is diagonalizable, find a diagonal matrix B and an invertible matrix Q such that $A = Q^{-1}BQ$, or such that $A = QBQ^{-1}$. Be sure to say which equation holds for your Q. (Note, you need not compute Q^{-1}.)

   The matrix in (3) is not diagonalizable. For the matrix A in (2), a basis of eigenvectors is $\beta = \{(1,1,0),(1,0,-1),(1,0,1)\}$, and the matrix of L_A in the basis β is the diagonal matrix B whose diagonal entries are the eigenvalues corresponding to those eigenvectors, $(1,1,-1)$. If Q is the matrix whose columns are the vectors in β, then Q changes from β-coordinates to standard coordinates, so we have $A = QBQ^{-1}$.

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}^{-1}.$$

   If you want to check your work without inverting Q (and maybe introducing new arithmetic errors), note that $A = QBQ^{-1}$, when multiplied by Q on the right, yields $AQ = QB$, which you can check by multiplying those matrices.

5. Can you always tell from the characteristic polynomial of a matrix whether that matrix is diagonalizable?

   No. The matrices in (2) and (3) have the same characteristic polynomial, but one is diagonalizable, and one is not.