Math 24
Spring 2012
Monday, May 7

(1.) TRUE or FALSE?

(a.) If E is an elementary matrix, then $det(E) = \pm 1$. (F)
(b.) For any $A, B \in M_{n\times n}(F)$, $det(AB) = (det(A))(det(B))$. (T)
(c.) A matrix $A \in M_{n\times n}(F)$ is invertible if and only if $det(A) = 0$. (F)
(d.) A matrix $A \in M_{n\times n}(F)$ has rank n if and only if $det(A) \neq 0$. (T)
(e.) For any $A \in M_{n\times n}(F)$, $det(A^t) = -det(A)$. (F)
(f.) The determinant of a square matrix can be evaluated by cofactor expansion along any column. (T)
(g.) Every system of n linear equations in n unknowns can be solved by Cramer's rule. (F)
(h.) Let $Ax = b$ be the matrix form of a system of n linear equations in n unknowns, where $x = (x_1, x_2, \ldots, x_n)^t$. If $det(A) \neq 0$ and if M_k is the $n \times n$ matrix obtained from A by replacing row k of A by b^t, then the unique solution of $Ax = b$ is

$$x_k = \frac{det(M_k)}{det(A)} \text{ for } k = 1, 2, \ldots, n.$$

(F)
(i.) If Q is an invertible matrix, then $det(Q^{-1}) = \dfrac{1}{det(Q)}$. (T)
(j.) The determinant of a lower triangular $n \times n$ matrix is the product of its diagonal entries. (A matrix is lower triangular if the only nonzero entries are on or below the main diagonal.)
(T)

(2.) Show that if A and B are similar $n \times n$ matrices, then $det(A) = det(B)$.

If $A = QBQ^{-1}$ then $det(A) = det(Q)det(B)det(Q^{-1}) = det(Q)det(B)\dfrac{1}{det(Q)} = det(B)$.

(3.) Suppose that $M \in M_{n\times n}(F)$ can be written in the form

$$M = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix},$$

where A is a square matrix, 0 is a zero matrix, and I is an $m \times m$ identity matrix. Prove that $det(M) = det(A)$.

One way to prove this is to use type 3 elementary row operations on the first $n - m$ rows of M, to put A (and therefore M) into upper triangular form, as

$$M^* = \begin{pmatrix} A^* & B^* \\ 0 & I \end{pmatrix}.$$

Then the diagonal entries of M^* are the diagonal entries of A^* and a bunch of 1's from I, and since their determinants are the product of their diagonal entries $det(M^*) = det)A^*)$. But type 3 elementary row operations don't change the determinant, and so $det(M) = det(M^*) = det(A^*) = det(A)$.

Another way to prove this is by induction on m. The base case is $m = 0$, in which case $M = A$, so $det(M) = det(A)$.

For the inductive step, assume this is true when $I = I_m$ and show it is true when $I = I_{m+1}$. In this case, displaying the last row and column of M, we have

$$M = \begin{pmatrix} A & B \\ 0 & I_{m+1} \end{pmatrix} = \begin{pmatrix} A & B^* & b^* \\ 0 & I_m & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and we can expand along the last row and use the inductive hypothesis to get

$$det(B) = (1)det \begin{pmatrix} A & B^* \\ 0 & I_m \end{pmatrix} = det(A).$$

(4.) Let $A \in M_{n\times n}(F)$ be nonzero. For any m with $1 \leq m \leq n$, an $m \times m$ submatrix is obtained by deleting $n - m$ rows and $n - m$ columns of A. For example, if we start with $A = \begin{pmatrix} 1 & 1 & 1 & 4 \\ 2 & 3 & 1 & 8 \\ -2 & 0 & 0 & -4 \\ 1 & 4 & 4 & 10 \end{pmatrix}$ and delete rows 2 and 3 and columns 2 and 4, we get the 2×2 submatrix $\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$.

(a.) Show that if A is an $n \times n$ matrix and there is a $k \times k$ submatrix of A with nonzero determinant, then $rank(A) \geq k$.

Since elementary row and column operations do not change the rank of a matrix, we can interchange rows and columns so the submatrix B with nonzero determinant sits in the upper left corner of A:

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}.$$

Since the first k-many columns of B are linearly independent, the first k-many columns of A must also be linearly independent. Therefore $rank(A) \geq k$.

(b.) Show that if A is an $n \times n$ matrix with rank k, then there is a $k \times k$ submatrix of A with nonzero determinant.

Since A has rank k, we can choose k linearly independent columns of A. Delete the rest to get an $n \times k$ matrix C of rank k.

Since C has rank k, we can choose k linearly independent rows of C. Delete the rest to get a $k \times k$ matrix B of rank k. Now B is a $k \times k$ submatrix of A, and since $rank(B) = k$, we know $det(B) \neq 0$.