Math 24
Spring, 2012
Wednesday, May 2, Sample Solutions

(1.) TRUE or FALSE?

(a.) If $(A'|b')$ is obtained from $(A|b)$ by a finite sequence of elementary column operations, then the systems $Ax = b$ and $A'x = b'$ are equivalent. (F)

(b.) If $(A'|b')$ is obtained from $(A|b)$ by a finite sequence of elementary row operations, then the systems $Ax = b$ and $A'x = b'$ are equivalent. (T)

(c.) If A is an $n \times n$ matrix with rank n, then the reduced row echelon form of A is I_n. (T)

(d.) Any matrix can be put in reduced row echelon form by means of a finite sequence of elementary row operations. (T)

(e.) If $(A|b)$ is in reduced row echelon form, then the system $Ax = b$ is consistent. (F)

(f.) Let $Ax = b$ be a system of m linear equations in n unknowns for which the augmented matrix is in reduced row echelon form. If this system is consistent, then the dimension of the solution set of $Ax = 0$ is $n - r$, where r equals the number of nonzero rows in A. (T)

(g.) If a matrix A is transformed by elementary row operations into a matrix A' in reduced row echelon form, then the number of nonzero rows in A' equals the rank of A. (T)

(2.) Use elementary row operations to convert the following matrix into reduced row echelon form:

$$\begin{pmatrix} 1 & 2 & 3 & -1 & 5 \\ 2 & 4 & 1 & -4 & 3 \\ 3 & 6 & 4 & -5 & 8 \\ 6 & 12 & 13 & -8 & 23 \end{pmatrix}$$

Unless I have messed up the arithmetic, this matrix converts to

$$\begin{pmatrix} 1 & 2 & 0 & -\frac{11}{5} & \frac{4}{5} \\ 0 & 0 & 1 & \frac{2}{5} & \frac{7}{5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(3.) Consider the following system of linear equations. Denote its coefficient matrix by A.

$$x_1 + 2x_2 + 3x_3 - x_4 = 5$$
$$2x_1 + 4x_2 + x_3 - 4x_4 = 3$$
$$3x_1 + 6x_2 + 4x_3 - 5x_4 = 8$$
$$6x_1 + 12x_2 + 13x_3 - 8x_4 = 23$$

Find the general solution to this system, the general solution to the corresponding homogenous system, and bases for the null space and range of L_A.

(Hint: Use your answer to problem (2).)

The augmented matrix of this system is the matrix of problem (2). We can use the reduced row echelon form to write down the equivalent system,

$$x_1 + 2x_2 - \frac{11}{5}x_4 = \frac{4}{5}$$
$$x_3 + \frac{2}{5}x_4 = \frac{7}{5}.$$

Using these equations to write x_1 and x_3 in terms of x_2 and x_4, and introducing parameters s for x_2 and t for x_4, we get the general solution,

$$(x_1, x_2, x_3, x_4) = s(-2, 1, 0, 0) + t\left(\frac{11}{5}, 0, -\frac{2}{5}, 1\right) + \left(\frac{4}{5}, 0, \frac{7}{5}, 0\right).$$

From this we can read off the general solution to the corresponding homogenous system,

$$(x_1, x_2, x_3, x_4) = s(-2, 1, 0, 0) + t\left(\frac{11}{5}, 0, -\frac{2}{5}, 1\right),$$

and a basis for the null space of L_A,

$$\left\{(-2, 1, 0, 0), \left(\frac{11}{5}, 0, -\frac{2}{5}, 1\right)\right\}.$$

Since the nullity of L_A is 2, the rank of L_A is $4 - 2 = 2$, so we can use any two linearly independent columns of the original coefficient matrix A as a basis for the range of L_A. To be systematic, we can look at the reduced row echelon form of A,

$$\begin{pmatrix} 1 & 2 & 0 & -\frac{11}{5} \\ 0 & 0 & 1 & \frac{2}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and note that columns 1 and 3 (e_1 and e_2) clearly form a basis for the span of the columns, so columns 1 and 3 of A form a basis for the span of the columns of A, which is the range of L_A:

$$\{(1, 2, 3, 6), (3, 1, 4, 13)\}.$$

(4.) Find the set of solutions for the homogeneous system associated to the following system of linear equations, and determine whether this system has a solution.

$$x + 2y - z = 1$$
$$2x + y + 2z = 3$$
$$x - 4y + 7z = 4$$

The augmented matrix of this system is

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 1 & 2 & 3 \\ 1 & -4 & 7 & 4 \end{pmatrix},$$

which row reduces to

$$\begin{pmatrix} 1 & 0 & \frac{5}{3} & \frac{5}{3} \\ 0 & 1 & -\frac{4}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The last equation of the system corresponding to this matrix is $0 = 1$, and therefore the original system has no solution.

However, from the reduced row echelon form of the coefficient matrix,

$$\begin{pmatrix} 1 & 0 & \frac{5}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 0 \end{pmatrix},$$

we can read off the general solution to the corresponding homogeneous system,

$$(x, y, z) = t\left(-\frac{5}{3}, \frac{4}{3}, 1\right).$$

(5.) Find the set of solutions for the following system of linear equations. Note that the coefficient matrix is the same as in problem (4), and one obvious solution is $x = 1$, $y = z = 0$.

$$x + 2y - z = 1$$
$$2x + y + 2z = 2$$
$$x - 4y + 7z = 1$$

We add our particular solution $(x, y, z) = (1, 0, 0)$ to the general solution to the homogeneous system, to get

$$(x, y, z) = t\left(-\frac{5}{3}, \frac{4}{3}, 1\right) + (1, 0, 0).$$

(6.) Prove or give a counterexample to the following statement: If the coefficient matrix of a system of m linear equations in n unknowns has rank m, then the system has a solution.

This statement is true.

Let A be the $m \times n$ coefficient matrix of the system, so the system is equivalent to a matrix equation $Ax = b$, which we can rewrite as $L_A(x) = b$. To show that $L_A(x) = b$ always has a solution, we must show that b is always in the range of L_A, that is, that L_A is onto.

Because A is $m \times n$, we know the codomain of L_A is F^m. Because the rank of A is m, we know the rank of L_A, or the dimension of the range of L_A, is m.

But now, since the dimension of the range equals the dimension of the codomain, L_A is onto. This is what we needed to show.

(7.) Let A be an $m \times n$ matrix with rank m. Prove that there exists an $n \times m$ matrix B such that $AB = I_m$. (Hint: Think about the linear transformation L_A.)

We know L_A is a linear transformation from F^n to F^m. If B is an $n \times m$ matrix, then AB is an $m \times m$ matrix, and L_{AB} is a linear transformation from F^m to F^m. We will have have $AB = I_m$ if we have $L_{AB} = I_{F^m}$. Since $L_{AB} = L_A L_B$, we must show there is a B such that $L_A L_B = I_{F^m}$.

Since any linear transformation T from F^m to F^n can be expressed $T = L_B$, where B is the $(n \times m)$ matrix of T relative to the standard bases, we need only show there is some linear transformation T from F^m to F^n such that $L_A T = I_{F^m}$. Since a linear transformation is completely determined by its action on the basis elements, it is enough to find T such that $L_A T(e_i) = e_i$ for $i = 1, 2, \ldots, m$.

We know that L_A is onto, since its rank is m. Therefore, for each i, we can choose a vector $v_i \in F^n$ such that $L_A(v_i) = e_i$. By a theorem from an earlier chapter, there is a linear transformation $T : F^m \to F^n$ such that, for each i, we have $T(e_i) = v_i$.

This T is the linear transformation we want: $L_A T(e_i) = L_A(T(e_i)) = L_A(v_i) = e_i$..Since $L_A T$ agrees with I_{F^n} on a basis for F_n, they are the same transformation. That is, $L_A T = I_{F^m}$, so if B is the matrix of T relative to the standard bases, then $I_{F^m} = L_A T = L_A L_B = L_{AB}$, and $AB = I_m$.