Math 24
Spring 2012
Thursday, April 19

(1.) TRUE or FALSE?

(a.) Suppose that $\beta = \{x_1, x_2, \ldots, x_n\}$ and $\beta' = \{x'_1, x'_2, \ldots, x'_n\}$ are ordered bases for a vector space and Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then the j^{th} column of Q is $[x_j]_{\beta'}$. (F, it's $[x'_j]_\beta$)

(b.) Every change of coordinate matrix is invertible. (T)

(c.) Let T be a linear operator on a finite-dimensional vector space V, let β and β' be ordered bases for V, and let Q be the change-of-coordinate matrix that changes β'-coordinates into β-coordinates. Then

$$[T]_\beta = Q[T]_{\beta'}Q^{-1}.$$

(T)

(d.) The matrices $A, B \in M_{n\times n}(F)$ are called similar if $B = Q^t AQ$ for some $Q \in M_{n\times n}(F)$. (F, it's $Q^{-1}AQ$)

(e.) Let T be a linear operator on a finite-dimensional vector space V. Then for any ordered bases β and γ for V, $[T]_\beta$ is similar to $[T]_\gamma$. (T)

(f.) Every invertible matrix is a change of coordinate matrix. (T)

(2.) Let $\alpha = \{(1,0),\ (0,1)\}$ and $\beta = \{(1,1),\ (1,-1)\}$ be ordered bases for $\mathbb{R}^2$.

(a.) Find the matrix A that changes β-coordinates into α-coordinates.

$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
Notice that the columns are α-coordinates of the basis vectors in β, that is, the standard coordinates of $(1,1)$ and $(1,-1)$.

(b.) Find the matrix B that changes α-coordinates into β-coordinates.

$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

(c.) Verify by matrix multiplication that $AB = I$.

Yes, it does.

(d.) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(1,1) = (2,2)$ and $T(1,-1) = (-1,1)$. Find the matrix $[T]_\beta$.

$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$
This is because $T(1,1) = 2(1,1) + 0(1,-1)$ and $T(1,-1) = 0(1,1) - 1(1,-1)$.

(e.) Compute $[T]_\alpha$ as a produce of matrices you have already found.

If Q changes β coordinates to α coordinates, then Q is the matrix in (a), Q^{-1} is the matrix in (b), and
$[T]_\alpha = Q[T]_\beta Q^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}$

(f.) Find $T(2,3)$.

$\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ \frac{9}{2} \end{pmatrix}$
$T(2,3) = (\frac{11}{2},\ \frac{9}{2})$.

(3.) For each linear transformation T, determine whether T is invertible and justify your answer.

(a.) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a, b) = (a - 2b, b, 3a + 4b)$.

(b.) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a, b) = (3a - 2, b, 4a)$.

(c.) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(a, b, c) = (3a - 2c, b, 3a + 4b)$.

(d.) $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(p(x)) = p'(x)$.

(e.) $T : M_{2\times2}(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c + d)x^2$.

(f.) $T : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + b & a \\ c & c + d \end{pmatrix}$.

Because an invertible linear transformation on an n-dimensional domain must have an n-dimensional codomain, we can see immediately that in (a), (b), and (d), T is not invertible.

If the dimensions of the domain and codomain are both n, to show T is invertible it is enough to show either that T is one-to-one or that T is onto.

In (e), T is not onto, because x^3 is not in its range, so T is not invertible.

In (c) and (f), T is one-to-one, so T is invertible; it is not hard to see in each case that the null space of T is $\{0\}$.

(4.) Let V be the subspace of $M_{2\times2}(F)$ spanned by $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Construct an isomorphism from V to F^3.

It is not hard to see this set is linearly independent, since none of these matrices is a linear combination of those listed earlier. They span all matrices of the form $\begin{pmatrix} a & a + b \\ 0 & c \end{pmatrix}$, which can also be phrased as all matrices of the form $\begin{pmatrix} a & d \\ 0 & c \end{pmatrix}$. This suggests two natural isomorphisms, $T\begin{pmatrix} a & a + b \\ 0 & c \end{pmatrix} = (a, b, c)$ and $T\begin{pmatrix} a & d \\ 0 & c \end{pmatrix} = (a, d, c)$. In each case it is easy to see that the range is all of F^3, and so T is an isomorphism.

There are three very important isomorphisms we have seen so far. We also know a few important facts about creating isomorphisms.

(a.) If V is any vector space, the identity transformation $I_V : V \to V$ is an isomorphism.

(b) If V is any n-dimensional vector space over F, and β is a basis for V, the "coordinate coding" function $\phi_\beta : V \to F^n$ defined by $\phi_\beta(v) = [v]_\beta$ is an isomorphism.

(c.) If V and W are n- and m- dimensional vector spaces over F with ordered bases α and β respectively, the "representation by matrix" function $\mathcal{R} : \mathcal{L}(V, W) \to M_{m\times n}(F)$ defined by $\mathcal{R}(T) = [T]_\alpha^\beta$ is an isomorphism.

(d.) If V and W are n-dimensional vector spaces over F, $\alpha = \{v_1, v_2, \ldots, v_n\}$ is a basis for V, and $\{w_1, w_2, \ldots, w_n\}$ is a basis for W, then there is a unique isomorphism $T : V \to W$ such that $T(v_1) = w_1, T(v_2) = w_2, \ldots T(v_n) = w_n$.

(e.) If $T : V \to W$ and $U : W \to Z$ are isomorphisms, the composite (or composition) $UT : V \to Z$ is also an isomorphism.

(f.) If $T : V \to W$ is an isomorphism, then T is invertible, and its inverse $T^{-1} : W \to V$ is also an isomorphism.

Make sure you understand these things. They will be useful.