Math 24 Exam 1 - Take Home portion

Directions: Your solutions to these exam problems are due at the beginning of class on Monday, 5 February 2001. You may feel free to use your book or your class notes from this course to help you with these problems, but **you may not use any other source of information. This includes books, papers, people and electronic devices.** The only exception is that you may ask your instructor for clarification.

Give complete and careful solutions to the problems. **Justify all conclusions**. You would be wise to quote any theorem you use by number or page number to make precise your reference (e.g., by Theorem 1.9 or by the Corollary on page 50...). Also, if you cannot prove one part of a problem, you may assume its validity to help you prove the next part of the problem. Last, but not least, neatness counts!

1. (10) Consider the system of linear equations with real coefficients.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 &= 0 \\ 3x_1 + 6x_2 + 10x_3 + 13x_4 + 17x_5 &= 0 \end{aligned}$$

   (a) Write down the matrix A so that the system above corresponds to the matrix equation $A\mathbf{x} = \mathbf{0}$. Find the dimension of and a basis for the rowspace of A.

   (b) Let W be the solution space to this system. Find a basis for W.

2. (10) Let $\{v_1, \ldots, v_n\}$ be a linearly independent subset of a vector space V, and let $w \in V$.

   (a) Show that if $\{v_1+w, v_2+w, \ldots, v_n+w\}$ is linearly dependent, then $w \in \text{Span}(\{v_1, \ldots, v_n\})$. **Hint:** If you don't use the linear independence of $\{v_1, \ldots, v_n\}$, your proof is wrong or incomplete.

   (b) Show that the converse of the above statement is false.

3. (10) Let V be a vector space of dimension $n \geq 2$. Let Z and W be subspaces with $\dim(Z) = r \geq 1$ and $\dim(W) = s \geq 1$, and such that $V = Z \bigoplus W$.

   (a) Let V' be a subspace of V and $Z' = Z \cap V'$ and $W' = W \cap V'$.
   Proof or counterexample: $V' = Z' \bigoplus W'$.

   (b) Show that each vector $v \in V$ can be written as $v = z + w$ for uniquely determined vectors $z \in Z$ and $w \in W$.

4. (20) Let $a \in \mathbb{R}$, and n a positive integer. Define

$$V(a) = \{f \in P_n(\mathbb{R}) \mid f(a) = 0\}$$

   (a) Show that $V(a)$ is a subspace of $P_n(\mathbb{R})$, and find a basis for $V(a)$.

   (b) Let W be the subspace $W = V(1) + V(2)$. Find $\dim(W)$ and a basis for W.