MATH 24
GROUP PROBLEMS 2
Due Monday, April 12 at the beginning of class

Group Members Names:____________________________________

Some common mistakes in the first homework assignment included:

- not starting sentences with words but with symbols. It's imperative that you tell the reader (me) what you're doing with the symbols.
- writing the proof up backwards. In the first problem, the one with the formula for $\sum_{i=1}^{n} i^2$ a lot of you started with the proposition and derived something like $0 = 0$. It's fine if you come up with the proof backwards, but you need to write it up forwards.
- any time you introduce a symbol you need to tell me what it is. In the second homework problem a lot of you in the inductive step said, "let n=2a+3b." Here it's important the $a, b \geq 0$: it wouldn't be a *sum* of 2's and 3's if one of them were negative. Not noticing this makes some of your proofs not one hundred percent correct.

This project is going to talk about the forward-backward method of proving things and a couple of different types of proof write-ups. You should compare what I'm doing with second common homework issue I described above.

Suppose I wanted to prove a statement of the form A implies B. For this example,

Proposition. *Let the right triangle $\triangle XYZ$ have side lengths of x and y, hypotenuse of length z and area $\frac{z^2}{4}$. Then $\triangle XYZ$ is isosceles.*

A is "right triangle $\triangle XYZ$ has its hypotenuse of length z and area $\frac{z^2}{4}$" and B is "$\triangle XYZ$ is isosceles." To prove this we can attack the problems from both ends. Let A_1 be a statement that is implied by A and B_1 be a statement that proves B.

There are many possible A_1's: using the part of A that says $\triangle XYZ$ is a triangle lets us say $\frac{1}{2}xy = \frac{z^2}{4}$; the fact that $\triangle XYZ$ is right, lets us say

$x^2+y^2=z^2$; the angles of $\triangle XYZ$ add to 180; etc. Similarly, we let A_i for $i \geq 2$ be a statement implied by A_{i-1}.

There are also many possible B_1's: if $x=y$, then $\triangle XYZ$ is isosceles; if one of the acute angles measures 45 degrees, then $\triangle XYZ$ is isosceles; etc. Similarly, we let B_i for $i \geq 2$ be statements that imply B_{i-1}.

How do you pick the right one of the many A_1's and B_1's? Sometimes it's luck, more often it's intuition, but most of the times it's common sense. If you ask the following **key question**: "How do I show $\triangle XYZ$ is isosceles?" and answer it with "Show $x=y$" then you see that you want to be using A_i's and B_i's that involve the lengths of the sides. You want to pick an A_1 that seems like it should lead to a statement like B. We're trying to show something about sides so we should choose an A_1 that involves sides. Similarly for B_1 we want to pick something that involves sides.

Here is a **full analysis** of one way to prove this proposition:

A right triangle $\triangle XYZ$ has its hypotenuse of length z and area $\frac{z^2}{4}$

B $\triangle XYZ$ is isosceles

We notice that our strongest (most specific) assumption is the one about area so we use that and make our A_1 an equality between two different ways of writing area. We also notice that we're going to be working with side lengths so we let our B_1 be $x=y$.

A: right triangle $\triangle XYZ$ has its hypotenuse of length z and area $\frac{z^2}{4}$

A_1: $\frac{1}{2}xy = \frac{z^2}{4}$

B_1: $x=y$

B: $\triangle XYZ$ is isosceles.

Now we note that in A_1 we have a statement that relates x, y and z. We make A_2 to be another such statement and hope that we can cancel. For B_2 we ask "How can you show two numbers are equal?" and note "if $x-y=0$ then $x=y$.

A: right triangle $\triangle XYZ$ has its hypotenuse of length z and area $\frac{z^2}{4}$

A_1: $\frac{1}{2}xy = \frac{z^2}{4}$

A_2: $x^2+y^2=z^2$

B_2: $x-y=0$

B_1: $x = y$

B: $\triangle XYZ$ is isosceles.

There isn't much more backwards work we can do so we finish with the following proof:

A: right triangle $\triangle XYZ$ has its hypotenuse of length z and area $\frac{z^2}{4}$

A_1: $\frac{1}{2}xy = \frac{z^2}{4}$

A_2: $x^2 + y^2 = z^2$

A_3: $\frac{1}{2}xy = \frac{x^2+y^2}{4}$

A_4: $x^2 - 2xy + y^2 = 0$

A_5: $(x - y)^2 = 0$

B_2: $x - y = 0$

B_1: $x = y$

B: $\triangle XYZ$ is isosceles.

Note that if you were a little more adventurous with the working backwards part that A_5 could easily have been B_3. You'll get better and more adventurous with this the more you practice.

A final (**homework level**) proof might look like

Proof. Let $\triangle XYZ$ be as in the proposition. We prove the proposition by showing $x = y$. The area of $\triangle XYZ$ is by assumption $\frac{z^2}{4}$ and by formula is $\frac{1}{2}xy$. Since the $\triangle XYZ$ is a right triangle, the Pythagorean Theorem tells us $z^2 = x^2 + y^2$. Plugging in this last equation and doing some algebra we see the following

$$\begin{aligned} \frac{z^2}{4} &= \frac{1}{2}xy \\ \frac{x^2+y^2}{4} &= \frac{1}{2}xy \\ x^2 + y^2 &= 2xy \\ (x-y)^2 &= 0. \end{aligned}$$

Since the only number whose square is 0, we see that $x - y = 0$ and conclude $x = y$. $\square$

Proofs in textbooks and research papers are very condensed as this following **succinct** proof shows:

Proof. The assumption and the Pythagorean theorem yield $x^2 + y^2 = 2xy$; hence $(x - y)^2 = 0$ and the triangle is isosceles as required. $\square$

Being able to prove things this succinctly is a right one earns. None of you should be proving things in this way yet.

Problems Do the following:

1. For each of the following **key questions**, list at least three answers:

   (a) How can I show two real numbers are equal?

   (b) How can I show two finite sets are equal?

2. From calculus. Suppose A is "the function f has a maximum at $x = c$." List at least two possible A_1's. Suppose now that B is "the function f has a maximum at $x = c$." List at least two possible B_1's.

3. Recall that $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is the vector space of functions that have real inputs and outputs. Let $\mathcal{C}^1(\mathbb{R})$ be the subset of differentiable functions in $\mathcal{F}(\mathbb{R}, \mathbb{R})$. Recall

   Definition. *A function $f : \mathbb{R} \to \mathbb{R}$ is* ***differentiable at a point*** *$c \in \mathbb{R}$ if*

   $$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

   is finite. If a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable for all $c \in \mathbb{R}$ we merely say f ***is differentiable****.*

   Show that $\mathcal{C}^1(\mathbb{R}) \leq \mathcal{F}(\mathbb{R}, \mathbb{R})$ by answering the following

   (a) What is the **key question** for this proof? Answer it.

   (b) Write out a **full analysis of the proof**.

   (c) Write out a **homework-level proof**.

   (d) Write out a **succinct proof**.