Systems of First Order ODEs, Part I

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Outline

Outline

The Definition

Definition

A system of first order ODEs is a system of equations of the form

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \ldots, x_n) \\ &\vdots \\ x_n' &= F_n(t, x_1, x_2, \ldots, x_n) \end{aligned} \tag{1.1}$$

where $x_1, \ldots, x_n$ are functions of the independent variable t. A solution of the system on the iterval $\alpha < t < \beta$ is a set of n functions

$$x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$$

that are differentiable on $\alpha < t < \beta$ and that satisfy the system for all $\alpha < t < \beta$.

Example

- Consider the system

$$\begin{aligned} x'(t) &= y(t) \\ y'(t) &= -x(t) \end{aligned}$$

- A solution is given by

$$(x(t), y(t)) = (\sin(t), \cos(t))$$

Example

- Consider the system

$$\begin{aligned} x'(t) &= -3x(t) + \sqrt{2}y(t) \\ y'(t) &= \sqrt{2}x(t) - 2y(t) \end{aligned}$$

- Every solution to this system is of the form

$$\Phi(t) = c_1(e^{-t}, \sqrt{2}e^{-t}) + c_2(-\sqrt{2}e^{-4t}, e^{-4t})$$

- **Note:** Solving systems like the one above will lead us to consider eigenvalues and eigenvectors.

Existence & Uniqueness

Theorem

In Eq. 1.1 suppose the functions $F_1, \ldots, F_n$ *and* $\partial F_1/\partial x_1, \ldots, \partial F_1/\partial x_n, \ldots, \partial F_n/\partial x_1, \ldots, \partial F_n/\partial x_n$ *are continuous in an open region* R *of the* $tx_1x_2\cdots x_n$*-space and let* $(t_0, x_1^0, \ldots, x_n^0) \in \mathbb{R}^{n+1}$*. Then there is an interval* $|t - t_0| < h$ *in which there exists a unique solution* $x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$ *of the system Eq.* 1.1 *such that*

$$(\phi_1(t_0), \ldots, \phi_n(t_0)) = (x_1^0, \ldots, x_n^0).$$

Linear System

Definition

The system Eq. 1.1 is said to be linear if the functions $F_1(t, x_1, \ldots, x_n), \ldots, F_n(t, x_1, \ldots, x_n)$ are linear in the variables $x_1, \ldots, x_n$. In which the system has the form

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \cdots p_{1n}(t)x_n + g_1(t) \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots p_{nn}(t)x_n + g_n(t) \end{aligned} \tag{1.2}$$

We'll say it is homogeneous in the case $g_1 = \cdots g_n = 0$. Otherwise it is said to be non-homogeneous.

Existence & Uniqueness

Theorem

If the functions $p_{ij}(t)$ $(1 \leq i, j \leq n)$ *and* $g_1, \dots, g_n$ *in Eq. 1.2 are continuous on an open interval* $I : \alpha < t < \beta$*, then there exists a unique solution* $x_1 = \phi_1(t), \dots, x_n = \phi_n(t)$ *of the system Eq.* 1.2 *defined on all of I such that*

$$(\phi_1(t_0), \dots, \phi_n(t_0)) = (x_1^0, \dots, x_n^0).$$

Spring-Mass System

Consider

1. Two blocks of mass m_1 and m_2.
2. Three springs with spring constants k_1, k_2 and k_3.
3. Assume an eternal force of $F_1(t)$ and $F_2(t)$ acting on the masses.

We get (using arguments similar to those used for the hanging block):

$$\begin{aligned} m_1 x_1'' &= -(k_1 + k_2)x_1 + k_2 x_2 + F_1(t) \\ m_2 x_2'' &= k_2 x_1 - (k_2 + k_3)x_2 + F_2(t) \end{aligned}$$

Spring-Mass System

4. Now make the substitution

$$y_1 = x_1, y_2 = x_2, y_3 = x_1' y_4 = x_2'$$

5. Then we get the following system of First-Order ODEs

$$\begin{aligned}
y_1' &= y_3 \\
y_2' &= y_4 \\
y_3' &= \frac{1}{m_1}(-(k_1 + k_2)y_1 + k_2 y_2 + F_1(t)) \\
y_4' &= \frac{1}{m_2}(k_2 y_1 - (k_2 + k_3)y_2 + F_2(t))
\end{aligned}$$

2nd Order Linear ODE to System of 1st Order ODEs

- Consider the 2nd Order ODE

$$u'' + p(t)u' + q(t)u = g(t)$$

- Let $x_1 = u$ and $x_2 = u'$.
- Then our ODE is equivalent to

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= -q(t)x_1(t) - p(t)x_2(t) + g(t) \end{aligned}$$

- **Moral:** Second-order linear ODEs are really just systems of First-Order linear ODEs.

Outline

$m \times n$ Matrices

An $m \times n$-matrix $A$is a rectangular array of complex numbers of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- The matrix is said to be square if $m = n$.
- A (column) vector is an $m \times 1$ matrix.

Matrix Operations

Definition

Let $A = (a_{ij})$ be an $m \times n$-matrix A

1. The transpose of A, denoted A^T, is the $n \times m$ matrix obtained by interchanging the rows and columns of A. Thus $A^t = (a_{ji})$.
2. The conjugate of A, denoted $\bar{A}$ is the $m \times n$ matrix obtained by replacing a_{ij} by its conjugate $\bar{a}_{ij}$. So, $\bar{A} = (\bar{a}_{ij})$.
3. The adjoint of A is given by $A^* = \bar{A}^t$.

New Matrices from Old

Let $A = \begin{pmatrix} 1+2i & 3 & 5-i \\ 0 & 2-9i & -4 \end{pmatrix}$, then

1. $A^t = \begin{pmatrix} 1+2i & 0 \\ 3 & 2-9i \\ 5-i & 0 \end{pmatrix}$
2. $\bar{A} = \begin{pmatrix} 1-2i & 3 & 5+i \\ 0 & 2+9i & -4 \end{pmatrix}$
3. $A^* = \begin{pmatrix} 1-2i & 0 \\ 3 & 2+9i \\ 5+i & -4 \end{pmatrix}$.

Matrix Addition & Multiplication

1. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$-matrices, then $A + B = (a_{ij} + b_{ij})$.
2. $\alpha \in \mathbb{C}$, then $\alpha A = (\alpha a_{ij})$
3. If $A = (a_{ij})$ is $m \times n$ and $B = (b_{ij})$ is $n \times r$, then AB is the $m \times r$ matrix $AB = (c_{ij})$ where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Inner Product

Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{C}^n$, be two vectors.

1. their inner product is the (complex) number

$$< x, y >= x^t \bar{y} = \sum_1^n x_i \bar{y}_i.$$

2. The length of x is given by

$$\|x\| =< x, x >^{\frac{1}{2}} .$$

3. x and y are said to be orthogonal if $< x, y >= 0$.

Inner Product

Example

Let $x = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $y = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$, then

- $\|x\| = \sqrt{11}$
- $x \perp y$.

The Determinant: the 2×2-case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix then the determinant is given by

$$\det(A) = ad - bc.$$

The Determinant: the 3×3-case

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a 3×3 matrix then the determinant is given by

$$\begin{aligned} \det(A) &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) \\ &\quad + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \end{aligned}$$

The Determinant: the General Case

Let $A = (a_{ij})$ be an $n \times n$ matrix

1. Let M_{ij} be the $(n-1) \times (n-1)$ matrix formed by deleting the i-th row and j-th column from A.
2. The determinant of A is the scalar given by

$$\det(A) = \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det(M_{1j}).$$

The Determinant: Exercises

Compute the determinant of the following matrices.

1. $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$
2. $\mathbf{B} = \begin{pmatrix} 2 & 3 \\ -2 & 7 \end{pmatrix}$
3. $\mathbf{C} = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$

Special Matrix: The Identity

The identity matrix I is the $n \times n$ matrix formed by placing ones down the diagonal and zeroes in all the other entries. So we have

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

If A is a square $n \times n$ matrix then

$$AI = IA = A.$$

The Inverse

Let A be a square $n \times n$ matrix. A is said to be invertible or non-singular if there exists an $n \times n$ matrix B such that

$$BA = AB = I.$$

If such a matrix exists it is unique and we denote it by A^{-1}.

Theorem

Let A be a square $n \times n$ matrix. A is invertible if and only if $\det(A) \neq 0$.

Computing A^{-1} via Cofactors

Let A be a square $n \times n$ matrix.

1. The cofactor associated to a_{ij} is

$$C_{ij} = (-1)^{i+j} \det M_{ij},$$

where M_{ij} is the $(n-1) \times (n-1)$ matrix formed by deleting the i-th row and j-th column.

2. If A is invertible, then

$$A^{-1} = (b_{ij}).,$$

where $b_{ij} = \frac{C_{ji}}{\det A}$.

Computing A^{-1} via Gaussian Elimination

- Elementary Row Operations
  1. interchange two rows
  2. multiply a row by a non-zero scalar
  3. adding any multiple of one row to another.
- Transforming A through a sequence of elem. row ops. is called Gaussian elimination or row reduction.
- A invertible, then there is a sequnce of elem. row ops. which transforms A to I and I to A^{-1}.

Computing A^{-1} via Gaussian Elimination

Example

Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. Use Gaussian elimination to show that

$$A^{-1} = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1/3 & -1/3 \\ 1 & 0 & -1 \end{pmatrix}.$$

Matrix Functions

Let $A(t) = (a_{ij}(t))$ and $B(t) = (b_{ij}(t))$ be matrix functions, then:

- $\frac{d}{dt}A(t) = (a'_{ij}(t))$
- If A and B are $n \times m$ then

$$\frac{d}{dt}(A(t) + B(t)) = A'(t) + B'(t).$$

- If A is $n \times r$ and B is $r \times m$, then

$$\frac{d}{dt}(A(t)B(t)) = A'(t)B(t) + A(t)B'(t).$$

The Definition

Definition

k vectors $\vec{x}_1, \ldots, \vec{x}_k \in \mathbb{R}^n$ are said to be linearly dependent if we can find numbers $c_1, \ldots, c_k \in \mathbb{R}$ not all zero such that

$$c_1\vec{x}_1 + \cdots + c_k\vec{x}_k = 0.$$

Otherwise we say the vectors are linearly independent.

Examples

1. $(1, 0)$ and $(0, 1)$ are linearly independent in $\mathbb{R}^2$.
2. $(1, 0, 0), (0, 1, 0)$ and $(1, 1, 0)$ are linearly dependent in $\mathbb{R}^3$.

A Test for Linear Independence

Theorem

Let $x^{(1)} = (x_{11}, \ldots, x_{n1}), \ldots x^{(n)} = (x_{1n}, \ldots, x_{nn})$ *be* n *vectors in* $\mathbb{R}^n$*, then they are linearly independent if and only if* $\det(\mathbf{X}) \neq 0$*, where* $\mathbf{X} = (x_{ij})$*.*

Example

Are the following sets of vectors lin. indep.?

1. $x^{(1)} = (1, 2, 3)$, $x^{(2)} = (1, -1, 0)$ and $x^{(3)} = (0, -1, 1)$
2. $x^{(1)} = (1, 0, 3)$, $x^{(2)} = (0, 1, 1)$ and $x^{(3)} = (-1, 3, 0)$

Systems & Matrices

A system of n equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

can be expressed as a matrix equation

$$\mathbf{Ax} = \mathbf{b}.$$

Systems & Matrices

Definition

The equation $\mathbf{Ax} = \mathbf{b}$ is said to be

- homogeneous, if $\mathbf{b} = \mathbf{0}$
- inhomogeneous if $\mathbf{b} \neq \mathbf{0}$

Theorem

Let $\mathbf{A}$ be an $n \times n$ matrix and $b \in \mathbb{R}^n$ a vector. Then

1. *$\mathbf{A}x = b$ has a unique solution if and only if $\det(\mathbf{A}) \neq 0$. In which case the solution is $x = \mathbf{A}^{-1}b$.*
2. *if $\det(\mathbf{A}) = 0$, then $\mathbf{A}x = b$ has no solutions or it has infinitely many solutions.*

Systems & Matrices

Let $\mathbf{A} = (a_{ij})$ be an $n \times n$ matrix

- Then we can think of $\mathbf{A}$ as a (linear) map $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^n$.
- To solve the equation $\mathbf{A}x = b$ means to find (all) $x \in \mathbb{R}^n$ such that $\mathbf{A}x = b \in \mathbb{R}^n$.

Eigenvectors & Eigenvalues

Consider the matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

- $\mathbf{A}$ scales the vector $\zeta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by a factor of 2
- (Actually, **A** scales any non-zero vector of the form $c \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by a factor of 2.)
- $\mathbf{A}$ scales the vector $\zeta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by a factor of $\frac{1}{2}$
- (Actually, **A** scales any non-zero vector of the form $c \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by a factor of $\frac{1}{2}$.)

Eigenvectors & Eigenvalues

Consider the matrix $\mathbf{B} = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}$

- $\mathbf{B}$ scales the vector $\zeta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by a factor of -3.
- (Actually, $\mathbf{B}$ scales any non-zero vector of the form $c \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by a factor of -3.)
- $\mathbf{B}$ fixes the vector $\zeta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
- (Actually, $\mathbf{B}$ fixes any non-zero vector of the form $c \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is fixed by $\mathbf{B}$.)

Eigenvectors & Eigenvalues

Consider the matrix $\mathbf{C} = \begin{pmatrix} 5 & 0 \\ -7 & 5 \end{pmatrix}$

- $\mathbf{C}$ scales the vector $\zeta_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by a factor of 5.
- (Actually, $\mathbf{C}$ scales any non-zero vector of the form $c\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by a factor of 5.)
- $\mathbf{C}$ does not scale in any other directions.

Eigenvectors & Eigenvalues

Consider the matrix $\mathbf{D} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

- $\mathbf{D}$ scales the vector $\zeta_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by a factor of 3.
- (Actually, $\mathbf{D}$ scales any non-zero vector of the form $c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by a factor of 3.)
- $\mathbf{D}$ scales the vector $\zeta_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ by a factor of -1.
- (Actually, $\mathbf{D}$ scales any non-zero vector of the form $c\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ by a factor of -1.)
- How did we know this?

Eigenvectors & Eigenvalues

- Given a matrix **A** it would be nice to find the non-zero vectors **x** such that
$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}.$$
- Equivalently, we want $\lambda \in \mathbb{C}$ and $\mathbf{x} \neq 0$ such that
$$\mathbf{A}\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}. \qquad (2.1)$$
- There is a non-zero solution **x** to Eq 2.1 if and only if $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.
- So, the first order of business is to find those values of λ such that
$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

Eigenvectors & Eigenvalues

Definition

Given an $n \times n$ matrix $\mathbf{A}$ its characteristic polynomial is the polynomial of degree n given by

$$\Delta(t) = \det(A - tI).$$

A root λ of $\Delta(t)$ is called an eigenvalue of the matrix A and a corresponding *non-zero* vector $\mathbf{x}$ such that

$$\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{0}$$

is an eigenvector corresponding to λ.

Eigenvectors & Eigenvalues

For a 2×2-matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ its characteristic polynomial is the degree 2 polynomial given by

$$\begin{aligned} \Delta(t) &= \det \begin{pmatrix} a-t & b \\ c & d-t \end{pmatrix} \\ &= t^2 - (a+b)t + (ad-bc) \\ &= t^2 - \mathrm{Tr}(\mathbf{A})t + \det \mathbf{A} \end{aligned}$$

Multiplicities

Definition

Let λ be an eigenvalue of an $n \times n$ matrix A. Then

1. the algebraic multiplicity of λ, denoted $\mathsf{mult}_{\text{alg}}(\lambda)$, is the number of times λ appears as a root of $\Delta(t)$.
2. the geometric multiplicity of λ, denoted $\mathsf{mult}_{\text{geom}}(\lambda)$, is the maximal number of linearly independent eigenvectors associated to λ.

We note that $\mathsf{mult}_{\text{geom}}(\lambda) \leq \mathsf{mult}_{\text{alg}}(\lambda) \leq n$.

Example I

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$$

- The characteristic polynomial of **A** is:

$$\Delta(\lambda) = \lambda^2 - 6\lambda + 8$$

- So the eigenvalues of **A** are $\lambda_1 = 2$ and $\lambda_2 = 4$.
- We now want to find the corresponding eigenvectors...

Example I (cont'd)

Case I: $\lambda_1 = 2$

- $\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \neq \mathbf{0}$ is an eigenvector corresponding to λ_1 if and only if

$$(\mathbf{A} - \lambda_1 I_2)\zeta = \mathbf{0}.$$

- This is equivalent to the system of equations

$$\begin{aligned} 3\zeta_1 - \zeta_2 &= 0 \\ 3\zeta_1 - \zeta_2 &= 0 \end{aligned}$$

- Hence, the eigenvectors are of the form

$$\zeta = c \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

for $c \neq 0$.

Example I (cont'd)

Case II: $\lambda_2 = 4$

- $\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \neq \mathbf{0}$ is an eigenvector corresponding to λ_2 if and only if
$$(\mathbf{A} - \lambda_2 I_2)\zeta = \mathbf{0}.$$
- This is equivalent to the system of equations
$$\begin{aligned} \zeta_1 - \zeta_2 &= 0 \\ 3\zeta_1 - 3\zeta_2 &= 0 \end{aligned}$$
- Hence, the eigenvectors are of the form
$$\zeta = c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
for $c \neq 0$.

Example II

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

- The characteristic polynomial of **A** is:

$$\Delta(\lambda) = -\lambda^3 + 6\lambda^2 + 15\lambda + 8$$

- The eigenvalues of **A** are $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 8$.
- We now want to find the corresponding eigenvectors...

Example II (cont'd)

Case I: $\lambda_1 = \lambda_2 = -1$

- $\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} \neq \mathbf{0}$ is an eigenvector corresponding to $\lambda_1 = \lambda_2$ if and only if

$$(\mathbf{A} - \lambda_1 I_2)\zeta = \mathbf{0}.$$

- This is equivalent to the system of equations

$$\begin{aligned} 4\zeta_1 + 2\zeta_2 + 4\zeta_3 &= 0 \\ 2\zeta_1 + \zeta_2 + 2\zeta_3 &= 0 \\ 4\zeta_1 + 2\zeta_2 + 4\zeta_3 &= 0 \end{aligned}$$

Example II (cont'd)

Case I: $\lambda_1 = \lambda_2 = -1$

- Hence, the eigenvectors corresponding to $\lambda_1 = \lambda_2 = -1$ are of the form

$$\zeta = c_1 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

  for c_1, c_2 not both zero.

- So, the eigenspace corresponding to the eigenvalue -1 is two-dimensional.

Example II (cont'd)

Case II: $\lambda_3 = 8$

- $\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} \neq \mathbf{0}$ is an eigenvector corresponding to $\lambda_1 = \lambda_2$ if and only if

$$(\mathbf{A} - \lambda_3 I_2)\zeta = \mathbf{0}.$$

- This is equivalent to the system of equations

$$\begin{aligned} -5\zeta_1 + 2\zeta_2 + 4\zeta_3 &= 0 \\ 2\zeta_1 - 8\zeta_2 + 2\zeta_3 &= 0 \\ 4\zeta_1 + 2\zeta_2 - 5\zeta_3 &= 0 \end{aligned}$$

Example II (cont'd)

Case II: $\lambda_3 = 8$

- Hence, the eigenvectors corresponding to $\lambda_3 = 8$ are of the form
$$\zeta = c \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$
for $c \neq 0$.
- So, the eigenspace corresponding to the eigenvalue 8 is one-dimensional.

Exercises

Find the eigenvalues and eigenvectors of the following matrices

1. $\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}$
2. $\mathbf{B} = \begin{pmatrix} \frac{7}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$
3. $\mathbf{C} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Outline

Examples

A general system of first-order linear ODEs is of the form

$$x'(t) = \mathbf{P}(t)x(t) + g(t),$$

where $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$, $g(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}$ and

$$\mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\ \vdots & \vdots & 0 & \vdots \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{pmatrix}.$$

Principle of Superposition

Proposition

Let $x^{(1)}(t), \ldots, x^{(n)}(t)$ *be vector functions which solve*

$$x' = \mathbf{P}(t)x.$$

Then for any $c_1, \ldots, c_n \in \mathbb{R}$ *we see that*

$$x(t) = c_1 x^{(1)}(t) + \cdots + c_n x^{(n)}(t)$$

also solves the equation.

The Wronskian

Let $x^{(1)}(t), \ldots, x^{(n)}(t)$ be vector functions, then the Wronskian is defined by

$$W[x^{(1)}, \ldots, x^{(n)}](t) = \det \mathbf{X}(t),$$

where

$$\mathbf{X}(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{pmatrix}$$

The Wronskian

Proposition

The vector functions $x^{(1)}(t), \ldots, x^{(n)}(t)$ *are linearly independent at* t_0 *if and only if*

$$W[x^{(1)}, \ldots, x^{(n)}](t_0) \neq 0.$$

The Wronskian

Proposition

Suppose $x^{(1)}(t), \dots, x^{(n)}(t)$ *are solutions to*

$$x' = \mathbf{P}(t)x, \ \alpha < t < \beta \tag{3.1}$$

such that $W[x^{(1)}, \dots, x^{(n)}](t) \neq 0$ *for all* $\alpha < t < \beta$. *Then for each* $x = \phi(t)$ *solving Eq. 3.1 there exist unique constants* $c_1, \dots, c_n \in \mathbb{R}$ *such that*

$$\phi(t) = c_1 x^{(1)}(t) + \cdots + c_n x^{(n)}(t).$$

In this case we call $\{x^{(1)}(t), \dots, x^{(n)}(t)\}$ *a fundamental set of solutions.*

The Wronskian

Proposition

If $x^{(1)}(t), \ldots, x^{(n)}(t)$ *are solutions to*

$$x' = \mathbf{P}(t)x, \ \alpha < t < \beta,$$

then on this interval $W[x^{(1)}, \ldots, x^{(n)}](t)$ *is either identically zero or never vanishes on* $\alpha < t < \beta$*.*

Corollary

A homogeneous linear system of the form

$$x' = \mathbf{P}(t)x, \ \alpha < t < \beta,$$

always has a fundamental set of solutions.