Second Order Linear ODEs, Part II

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Outline

1. Non-homogeneous Linear Equations
2. Method of Undetermined Coefficients
   - Motivating Examples
   - What's going on?
   - Exercises
3. Variation of Parameters
4. Applications
   - Mechanical Vibration
   - Undamped & Damped Free Vibration
   - Forced Vibrations

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The Idea

Question

How do we find the general solutions to a non-homogeneous 2nd order linear ODE

$$y'' + p(t)y' + q(t)y = g(t)?$$

We recall that there is an associated homogeneous equation

$$y'' + p(t)y' + q(t)y = 0.$$

The Idea

Theorem

Let $Y_1(t)$ and $Y_2(t)$ be two solutions to the non-homogeneous linear equation

$$y'' + p(t)y' + q(t)y = g(t), \tag{1.1}$$

then $Y_1 - Y_2$ solves the corresponding homogeneous equation. Hence, the general solution to Eq. 1.1 is of the form

$$\phi(t) = Y(t) + c_1 y_1(t) + c_2 y_2(t),$$

where Y is some solution to Eq. 1.1 and y_1 and y_2 form a fundamental set of solutions for the corresponding homogeneous equation.

The Idea

So our strategy for solving

$$y'' + p(t)y' + q(t)y = g(t)$$

is:

1. Find some solution $Y(t)$ to the non-homogeneous equation.
2. Find the general solution $c_1 y_1(t) + c_2 y_2(t)$ of the associated homogeneous equation.
3. Then $Y(t) + c_1 y_1(t) + c_2 y_2(t)$ is the general solution.

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Motivating Example I

Consider the equation

$$y'' - 5y' + 6y = 3\sin(t).$$

Step 1: Find a solution $Y(t)$

- Since RHS involves trig functions we assume

$$Y(t) = A\cos(t) + B\sin(t)$$

.

- Then $Y'(t) = -A\sin(t) + B\cos(t)$ and $Y''(t) = -A\cos(t) - B\sin(t)$.
- Substitute to get system:

$$\begin{aligned} 5A - 5B &= 0 \\ 5A + 5B &= 3 \end{aligned}$$

- $A = \frac{3}{10}$ and $B = \frac{3}{10}$.

Motivating Example I

Step 2: Fundamental Set

- Corresponding homogeneous equation $y'' - 5y' + 6y = 0$.
- General solution of homog. eq. is

$$c_1 e^{-3t} + c_2 e^{-2t}.$$

Step 3: General solution non-homog. eq. is given by

$$\phi(t) = \frac{3}{10}\cos(t) + \frac{3}{10}\sin(t) + c_1 e^{-3t} + c_2 e^{-2t}.$$

Motivating Example I

Moral

By taking our lead from the RHS of the equation

$$y'' - 5y' + 6y = 3\sin(t)$$

and assuming $Y = A\cos(t) + B\sin(t)$ we found the general solution to our problem.

Motivating Example II

Consider the equation

$$y'' - 3y' - 4y = 3e^{2t}.$$

Step 1: Find a solution $Y(t)$

- Since RHS involves an exponential assume $Y(t) = Ae^{2t}$.
- Then $Y'(t) = 2Ae^{2t}$ and $Y''(t) = 4Ae^{2t}$.
- Substituting we get $-6Ae^{2t} = 3e^{2t}$.
- Hence, $A = -\frac{1}{2}$ and

$$Y(t) = -\frac{1}{2}e^{2t}.$$

Motivating Example II

Step 2: Fundamental Set

- Corresponding homogeneous equation $y'' - 3y' - 4y = 0$.
- General solution of homog. eq. is

$$c_1 e^{-1t} + c_2 e^{4t}.$$

Step 3: General solution non-homog. eq. is given by

$$\phi(t) = -\frac{1}{2}e^{2t} + c_1 e^{-t} + c_2 e^{4t}.$$

Motivating Example II

Moral

By taking our lead from the RHS of the equation

$$y'' - 3y' - 4y = 3e^{2t}$$

and assuming $Y = Ae^{2t}$ we found the general solution to our problem.

Motivating Example III

Consider the equation

$$y'' - 3y' - 4y = 2e^{-t}.$$

Step 1: Find a solution $Y(t)$.

- Since RHS involves an eponential assume $\tilde{Y}(t) = Ae^{-t}$.
- But Ae^{-t} solves homogeneous equation. Hmm...
- Assume $Y(t) = Ate^{-t}$
- Then $Y'(t) = Ae^{-t} - Ate^{-t}$ and $Y''(t) = -2Ae^{-t} + Ate^{-t}$.
- Substituting we get $A = -\frac{2}{5}$.
- Hence, $Y(t) = -\frac{2}{5}te^{-t}$ solves our equation.

Motivating Example III

Step 2: Fundamental Set

- Corresponding homogeneous equation $y'' - 3y' - 4y = 0$.
- General solution o homog. eq. is

$$c_1 e^{-1t} + c_2 e^{4t}.$$

Step 3: General solution non-homog. eq. is given by

$$\phi(t) = -\frac{2}{5} t e^{-t} + c_1 e^{-t} + c_2 e^{4t}.$$

Motivating Example IV

Consider the equation

$$y'' + 3y' + y = t^3 + 3t + 5.$$

Step 1: Find a solution $Y(t)$.

- Since RHS involves a polynomial assume $Y(t) = A_3 t^3 + A_2 t^2 + A_1 t + A_0$.
- Then $Y'(t) = 3A_3 t^2 + sA_2 t + A_1$ and $Y''(t) = 6A_3 t + 2A_2$.
- Substituting we conclude

$$A_0 = -130, A_1 = 51, A_2 = -9, A_3 = 1.$$

- Hence, $Y(t) = t^3 - 9t^2 + 51t - 130$ solves our equation.

Motivating Example IV

Step 2: Fundamental Set

- Corresponding homogeneous equation $y'' + 3y' + y = 0$.
- General solution to homog. eq. is

$$c_1 e^{\frac{-3+\sqrt{5}}{2}t} + c_2 e^{\frac{-3-\sqrt{5}}{2}t}.$$

Step 3: General Solution to non-homog. eq. is given by

$$\phi(t) = t^3 - 9t^2 + 51t - 130 + c_1 e^{\frac{-3+\sqrt{5}}{2}t} + c_2 e^{\frac{-3-\sqrt{5}}{2}t}.$$

Solving $ay'' + by' + cy = P_n(t)$

- Let $P_n(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$.
- To find a solution of $ay'' + by' + cy = P_n(t)$, our candidate is of the form

$$Y(t) = t^s (A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0),$$

where s equals the number of times 0 is a root of the characteristic equation $ax^2 + bx + c$.

Solving $ay'' + by' + cy = P_n(t)$

- Consider $3y'' - 2y' = t + 5$
- The RHS is a polynomial
- Since 0 is a single root of $3x^2 - 2x$, our candidate is of the form
$$Y(t) = t^1(A_1 t + A_0) = A_1 t^2 + A_0 t$$
- Substituting we find $A_1 = -\frac{1}{4}$ and $A_0 = -\frac{13}{4}$, and we conclude that
$$Y(t) = -\frac{1}{4}t^2 - \frac{13}{4}t$$
is a solution to our ODE. What is the general solution?

Solving $ay'' + by' + cy = P_n(t)e^{\alpha t}$

- As before, let $P_n(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$
- Let α be some real constant.
- To solve $ay'' + by' + cy = P_n(t)e^{\alpha t}$, our candidate is of the form

$$Y(t) = t^s(A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0)e^{\alpha t},$$

where s equals the number of times α is a root of the characteristic equation $ax^2 + bx + c$.

Solving $ay'' + by' + cy = P_n(t)e^{\alpha t}$

- Consider $y'' - 6y' + 9y = e^{3t}$
- the RHS is $1e^{3t}$
- Since 3 is a double root of $x^2 - 6x + 9$, our candidate is of the form
$$Y(t) = t^2 A_0 e^{3t}$$
- Substituting we find $A_0 = \frac{1}{2}$ and we conclude
$$Y(t) = \frac{1}{2}t^2 e^{3t}$$
is a solution to our ODE. What is the general solution?

Solving $ay'' + by' + cy = P_n(t)e^{\alpha t}\cos(\beta t)$

- As before, let $P_n(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$
- Let α and β be some real constant.
- To solve $ay'' + by' + cy = P_n(t)e^{\alpha t}\cos(\beta t)$, our candidate is of the form

$$\begin{aligned} Y(t) &= t^s(A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0)e^{\alpha t}\cos(\beta t) \\ &\quad + t^s(B_n t^n + B_{n-1}t^{n-1} + \cdots + B_1 t + B_0)e^{\alpha t}\sin(\beta t) \end{aligned}$$

where s equals the number of times $\alpha + i\beta$ is a root of the characteristic equation $ax^2 + bx + c$.

Solving $ay'' + by' + cy = P_n(t)e^{\alpha t}\cos(\beta t)$

- Consider $y'' + 4y = \cos(2t)$
- The RHS is $1e^{0t}\cos(2t)$.
- Since $0 + i2$ is a single root of $x^2 + 4$, our candidate is of the form

$$Y(t) = t^1 e^{0t}(A_0\cos(2t) + B_0\sin(2t)) = t(A_0\cos(2t) + B_0\sin(2t))$$

- Substituting we find $A_0 = 0$ and $B_0 = \frac{1}{4}$ and we conclude

$$Y(t) = \frac{1}{4}t\sin(2t)$$

is a solution to our ODE. What is the general solution?

Solving $ay'' + by' + cy = P_n(t)e^{\alpha t}\sin(\beta t)$

- As before, let $P_n(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$
- Let α and β be some real constant.
- To solve $ay'' + by' + cy = P_n(t)e^{\alpha t}\sin(\beta t)$, our candidate is of the form

$$\begin{aligned} Y(t) &= t^s(A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0)e^{\alpha t}\cos(\beta t) \\ &\quad + t^s(B_n t^n + B_{n-1} t^{n-1} + \cdots + B_1 t + B_0)e^{\alpha t}\sin(\beta t) \end{aligned}$$

where s equals the number of times $\alpha + i\beta$ is a root of the characteristic equation $ax^2 + bx + c$.

Solving $ay'' + by' + cy = P_n(t)e^{\alpha t}\sin(\beta t)$

- Consider $y'' + y = e^t \sin(2t)$
- The RHS is $1e^{1t}\sin(2t)$.
- Since $1 + i2$ is not a root of $x^2 + 1$, our candidate is of the form

$$Y(t) = t^0 e^{1t}(A_0\cos(2t) + B_0\sin(2t)) = e^t(A_0\cos(2t) + B_0\sin(2t))$$

- Substituting we find $A_0 = -\frac{1}{5}$ and $B_0 = -\frac{1}{10}$ and we conclude

$$Y(t) = -e^t\left(\frac{1}{5}\cos(2t) + \frac{1}{10}\sin(2t)\right)$$

is a solution to our ODE. What is the general solution?

The Technique

To solve $ay'' + by' + cy = g(t)$.

1. Find fund. set of sol. $\{y_1, y_2\}$ to homogeneous eq.
2. Check that $g(t)$ involves only polynomials, exponentials, sines & cosines, and sums & products of the above.
3. If $g(t) = g_1(t) + \cdots + g_n(t)$ set up n subproblems:

$$ay'' + by' + cy = g_j(t), \; j = 1, \ldots, n.$$

4. The form of g_j and the roots of $ax^2 + bx + c$ determine the form of our candidate solution $Y_j(t)$ to the above.
5. Now solve for Y_j in each subproblem.
6. $Y(t) = Y_1(t) + \cdots + Y_n(t)$ and general solution is

$$\phi(t) = Y(t) + c_1 y_1(t) + c_2 y_2(t).$$

Exercises

1. Find a solution to the following differential equations
   1. $y'' + 3y' + y = t^3 + 3t + 5$;
   2. $y'' + 3y' = t^3 + 3t + 5$;
   3. $y'' = t^3 + 3t + 5$;

   Note: How did the form of your "guess" change in each of the above?
2. Find the general solution to
$$2y'' + 3y' + y = t^2 + 3\sin(t).$$
3. Find the general solution to
$$y'' + 8y' + 16y = e^{-4t}.$$

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Motivating Example

Consider the ODE

$$y'' - 5y' + 6y = 2e^t.$$

- $c_1 e^{3t} + c_2 e^{2t}$ solve hom. eq.
- Assume $Y(t) = u_1(t)e^{3t} + u_2(t)e^{2t}$.
- Then

$$Y'(t) = 3u_1 e^{3t} + 2u_2 e^{2t}$$

  if we assume $u_1' e^{3t} + u_2' e^{2t} = 0$.
- Then $Y'' = 9u_1 e^{3t} + 4u_2 e^{2t} + 3u_1' e^{3t} + 2u_2' e^{2t}$.
- Substitute to get

$$3u_1' e^{3t} + 2u_2' e^{2t} = 2e^t.$$

Motivating Example

- So we get the system

$$\begin{aligned} u_1' e^{3t} + u_2' e^{2t} &= 0 \\ 3u_1' e^{3t} + 2u_2' e^{2t} &= 2e^t \end{aligned}$$

- Solving we get

$$u_1'(t) = 2e^{-2t} \text{ and } u_2'(t) = -2e^{-t}.$$

- $u_1(t) = -e^{-2t} + c_1$ and $u_2(t) = 2e^{-t} + c_2$.
- $Y(t) = u_1(t)e^{3t} + u_2(t)e^{2t} = e^t + c_1 e^{3t} + c_2 e^{2t}$.

The Method

Theorem (Variation of Parameters)

Let p, q, g be cont. on I and if y_1 and y_2 are a fund. set of sols. to the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

then

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)}ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)}ds$$

solves the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t).$$

Exercises

1. Use variation of parameters to solve
$$y'' + 2y' + y = 3e^{-t}.$$
2. Use variation of parameters to solve
$$y'' + 4y = t^2 + 7.$$

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Spring-Mass System: The Set-up

- Consider a mass m hanging from the end of a vertical spring of length ℓ.
- The mass causes the spring to stretch L units in the downward (positive) direction.
- Two Forces acting on the mass
  1. Gravity: $+mg$
  2. Restoring Force of spring: $F_s = -kL$ (Hooke's law)
- Spring in equilibrium: $mg - kL = 0$
- Now let $u(t)$ denote the displacement of the mass from its equilibrium position.

Spring-Mass System: The Set-up

- Newton's law states

$$mu''(t) = F(t),$$

  where $F(t)$ is the sum of forces acting on the mass at time t.
- What are the forces acting on the mass?
  1. Gravity: mg;
  2. Spring Force: $F_s = -k(L + u(t))$ (Hooke's law);
  3. Damping Force: $F_d = -\gamma u'(t)$, $\gamma > 0$;
  4. An applied external force: $F_e(t)$.

Spring-Mass System: The Set-up

- So we obtain a 2nd Order linear ODE:

$$
\begin{aligned}
mu''(t) &= mg + F_s(t) + F_d(t) + F_e(t) \\
&= mg - k(L + u(t)) - \gamma u'(t) + F_e(t) \\
&= -ku(t) - \gamma u'(t) + F_e(t)
\end{aligned}
$$

  since $mg - kL = 0$.
- By Existence and uniqueness theorem there is a unique solution to the IVP

$$
mu''(t) + \gamma u'(t) + ku(t) = F_e(t), \; u(t_0) = u_0, u'(t_0) = v_0.
$$

- Physical interpretation of Exist. & Uniqueness: if we do an experiment repeatedly with the exact same initial conditions we will get the same result each time.

Spring-Mass System & Undamped Free Vibration

- Consider a spring-mass system where $\gamma = 0$ and $F_e = 0$.
- Then we get
$$mu''(t) + ku(t) = 0.$$
- The general solution is
$$u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t),$$
where $\omega = \frac{k}{m}$.
- Can be expressed (using double angle formula) as
$$u(t) = R\cos(\omega_0 t - \delta),$$
where $A = R\cos(\delta)$ and $B = R\sin(\delta)$.

Spring-Mass System & Undamped Free Vibration

Definition

Consider the spring-mass system with undamped free vibration.

1. $\omega_0 = \sqrt{\frac{k}{m}}$ is the **natural frequency** of the vibration (measured in radians per unit time).
2. The **period** of the motion is $T = \frac{2\pi}{\omega_0}$. It measures the amount of time between successive peaks of the graph.
3. R is the **amplitude** of the motion;
4. δ is called the **phase**. It measures the displacement of the wave with respect to its usual position.

Spring-Mass System & Undamped Free Vibration

Consider the spring-mass system governed by

$$3u'' + 2u = 0.$$

1. Find the general solution to the ODE
2. express your solution as $u(t) = R\cos(\omega_0 t - \delta)$
3. Sketch a graph of your solution
4. How much time passes between successive maxima?
5. How many radians are swept out in this period?
6. What is the maximum displacement of the mass from equilibrium?
7. Describe the long-run behavior

Spring-Mass System & Damped Free Vibration

Consider the spring-mass system governed by

$$3u'' + \gamma u' + 2u = 0, \ \gamma > 0$$

1. Find the general solutions to this ODE. (there will be three cases).
2. What can you say about long-run behavior of these solutions?
3. Of the solutions you came up with, which seems closest to periodic.
4. Express this quasi-periodic solution in the form $u(t) = Re^{-\alpha t}\cos(\mu t - \delta)$
5. Sketch a graph of your solution.

Spring-Mass System & Damped Free Vibration

Definition

Consider the spring-mass system with damped free vibration: $mu'' + \gamma u' + ku = 0,\ \gamma > 0.$

1. $\mu = \frac{\sqrt{4km - \gamma^2}}{2m}$ is the **quasi-frequency** of the vibration (measured in radians per unit time).
2. The **quasi-period** of the motion is $T_d = \frac{2\pi}{\mu}$.

Small Damping, Critical Damping & Overdamping

Definition

Consider the spring-mass system with damped free vibration: $mu'' + \gamma u' + ku = 0, \ \gamma > 0$.

1. When $0 < \gamma < 2\sqrt{km}$, the solution is of the form:
$$u(t) = Re^{-\gamma t/2m}\cos(\mu t - \delta);$$
2. When $\gamma = 2\sqrt{km}$ this is called **critical damping** and the solution is of the form
$$u(t) = (A + tB)e^{-\gamma t/2m}.$$
3. When $\gamma > 2\sqrt{km}$ this is called **overdamping** and the solution is of the form
$$u(t) = Ae^{r_1 t} + Be^{r_2 t}.$$

Small Damping, Critical Damping & Overdamping

Moral

In each of the cases the solutions die out in the limit. For this reason these solutions are sometimes called **transient solutions**.

Spring-Mass System with Damping & External Force

- We recall that the general spring-mass system is modeled by the ODE
$$mu'' + \gamma u' + ku = F_e(t),$$
where $m, \gamma, k > 0$.
- Suppose $F_e(t) = F_0 \cos(\omega t)$, then the general solution looks like
$$u(t) = A\cos(\omega t) + B\sin(\omega t) + c_1 u_1(t) + c_2 u_2(t),$$
where u_1, u_2 solves the homogeneous equation.
- Let $u_c(t) \equiv c_1 u_1(t) + c_2 u_2(t)$, $U(t) = A\cos(\omega t) + B\sin(\omega t)$.

Spring-Mass System with Damping & External Force

- By previous discussion $u_c(t) \equiv c_1 u_1(t) + c_2 u_2(t)$ dies off as $t \to \infty$:

$$\lim_{t \to \infty} |u_c(t)| = 0.$$

  It is transient.

- $U(t) = A\cos(\omega t) + B\sin(\omega t)$ is the steady state solution.