2×2 Matrices & Systems of Linear Equations

Craig J. Sutton
craig.j.sutton@dartmouth.edu

Department of Mathematics
Dartmouth College

Math 23 Differential Equations
Winter 2013

Outline

Outline

2×2 Matrices

Definition

A **2×2-matrix** is a 2×2 array of numbers of the form:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where the a_{ij}'s are real numbers. For example,

$$A = \begin{bmatrix} 2 & 3 \\ -4 & 7 \end{bmatrix}$$

is a 2×2-matrix.

Multiplication

Definition

Given two matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ we can multiply them to get another 2×2-matrix as follows.

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}.$$

Multiplication Example

Example

Let $A = \begin{bmatrix} 4 & 3 \\ -2 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 1 & 3 \\ 25 & 7 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} -8 & -6 \\ 10 & 16 \end{bmatrix}.$$

Hence, we see that in general $AB \neq BA$. So matrix multiplication is not commutative, the way normal (scalar) multiplication is.

Multiplication Example

Example

Let $A = \begin{bmatrix} 4 & 3 \\ -2 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 1 & 3 \\ 25 & 7 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} -8 & -6 \\ 10 & 16 \end{bmatrix}.$$

Hence, we see that in general $AB \neq BA$. So matrix multiplication is not commutative, the way normal (scalar) multiplication is.

Outline

The Determinant

Associated to any 2×2-matrix A is a special number known as the determinant.

Definition

Given any 2×2-matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ we define the **determinant** as

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

The Determinant

- Consider the 2×2-matrix $A = \begin{bmatrix} 4 & -2 \\ 7 & 1 \end{bmatrix}$
- Then $\det(A) = 18$.

Theorem

If A and B are 2×2-matrices, then $\det(AB) = \det(A) \cdot \det(B)$.

The Determinant

- Consider the 2×2-matrix $A = \begin{bmatrix} 4 & -2 \\ 7 & 1 \end{bmatrix}$
- Then $\det(A) = 18$.

Theorem

If A and B are 2×2-matrices, then $\det(AB) = \det(A) \cdot \det(B)$.

The Identity Matrix

Definition

A special 2×2-matrix is the so-called **identity matrix**

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The identity matrix I has the unique property that given any 2×2-matrix A we have

$$IA = A \text{ and } AI = A.$$

So, I acts on matrices the way the number 1 acts on other scalars (i.e., $1 \cdot a = a$ and $a \cdot 1 = a$).

The Identity Matrix

Definition

A special 2×2-matrix is the so-called **identity matrix**

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The identity matrix I has the unique property that given any 2×2-matrix A we have

$$IA = A \text{ and } AI = A.$$

So, I acts on matrices the way the number 1 acts on other scalars (i.e., $1 \cdot a = a$ and $a \cdot 1 = a$).

The Identity Matrix

Definition

A special 2×2-matrix is the so-called **identity matrix**

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The identity matrix I has the unique property that given any 2×2-matrix A we have

$$IA = A \text{ and } AI = A.$$

So, I acts on matrices the way the number 1 acts on other scalars (i.e., $1 \cdot a = a$ and $a \cdot 1 = a$).

Invertible Matrices

Definition

A matrix A is said to be **invertible** or **non-singular** if there is a matrix B such that

$$BA = I \text{ and } AB = I.$$

In this case the matrix B is unique and we denote it by A^{-1}.

Example

Let $A = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}$, then one can check that $\begin{bmatrix} \frac{3}{15} & -\frac{1}{15} \\ 0 & \frac{5}{15} \end{bmatrix}$ is its inverse.

Invertible Matrices

Definition

A matrix A is said to be **invertible** or **non-singular** if there is a matrix B such that

$$BA = I \text{ and } AB = I.$$

In this case the matrix B is unique and we denote it by A^{-1}.

Example

Let $A = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}$, then one can check that $\begin{bmatrix} \frac{3}{15} & -\frac{1}{15} \\ 0 & \frac{5}{15} \end{bmatrix}$ is its inverse.

Test for Invertibility

We now explain the significance of the determinant.

Theorem

A 2×2*-matrix* A *is invertible if and only if* $\det(A) \neq 0$*. In this case*

$$A^{-1} = \begin{bmatrix} \frac{a_{22}}{\det(A)} & -\frac{a_{12}}{\det(A)} \\ -\frac{a_{21}}{\det(A)} & \frac{a_{11}}{\det(A)} \end{bmatrix}.$$

Test for Invertibility

Example

Let $A = \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}$, then $\det(A) = 8$, so A is invertible and its inverse is

$$A^{-1} = \begin{bmatrix} 5/8 & 2/8 \\ 1/8 & 2/8 \end{bmatrix}.$$

Outline

1. The Definition & Matrix Multiplication
2. The Determinant & Invertibility
3. Matrices & Systems of Linear Equations

Given a 2×2-matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and a 2×1-matrix $b = \begin{bmatrix} x \\ y \end{bmatrix}$ we may define their product to give a 2×1-matrix as follows.

$$\begin{aligned} Ab &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix} \end{aligned}$$

Example

Let $A = \begin{bmatrix} -1 & 0 \\ 6 & -3 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, then

$$Ab = \begin{bmatrix} -4 \\ 27 \end{bmatrix}.$$

System of Linear Equations

Recall from High School the following problem

Problem

Find all pairs of numbers (x, y) such that

$$\begin{aligned} a_{11}x + a_{12}y &= b_1 \\ a_{21}x + a_{22}y &= b_2 \end{aligned}$$

Each equation represents a line in the xy-plane. So we're looking for all intersections of these two lines.

System of Linear Equations

Recall from High School the following problem

Problem

Find all pairs of numbers (x, y) such that

$$\begin{aligned} a_{11}x + a_{12}y &= b_1 \\ a_{21}x + a_{22}y &= b_2 \end{aligned}$$

Each equation represents a line in the *xy*-plane. So we're looking for all intersections of these two lines.

System of Linear Equations

Given two lines in the *xy*-plane we have the following possibilities.

1. The lines are parallel
    - Lines don't intersect: no solutions to the system, or
    - the lines are precisely the same: infinitely many solutions to the system.
2. The lines aren't parallel
    - exactly one solution to the system.

System of Linear Equations

Given two lines in the *xy*-plane we have the following possibilities.

1. The lines are parallel
   - Lines don't intersect: no solutions to the system, or
   - the lines are precisely the same: infinitely many solutions to the system.
2. The lines aren't parallel
   - exactly one solution to the system.

System of Linear Equations

Given two lines in the *xy*-plane we have the following possibilities.

1. The lines are parallel
   - Lines don't intersect: no solutions to the system, or
   - the lines are precisely the same: infinitely many solutions to the system.
2. The lines aren't parallel
   - exactly one solution to the system.

System of Linear Equations

The system

$$\begin{aligned} a_{11}x + a_{12}y &= b_1 \\ a_{21}x + a_{22}y &= b_2 \end{aligned}$$

can be expressed in matrix form by

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

System of Linear Equations

Theorem

The system

$$\begin{aligned} a_{11}x + a_{12}y &= b_1 \\ a_{21}x + a_{22}y &= b_2 \end{aligned}$$

has a unique solution if and only if $\det(A) \neq 0$. *In which case the unique solution is given by*

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Examples

Determine whether the following systems have unique solutions or not. If unique, find the solution.

1. $$\begin{aligned} 5x + 2y &= 7 \\ -1x + 2y &= -1 \end{aligned}$$

2. $$\begin{aligned} 3x + 2y &= 5/2 \\ -6x + -4y &= 4 \end{aligned}$$

System of Linear Equations

Moral

- Systems of linear equations can also be expressed using matrix notation.
- The determinant gives us a test for whether a system has a unique solution or not.
- Non-zero determinant allows us to easily find the unique solution of the system.
- A zero determinant tells us the system either has no solutions or it has infinitely many solutions.

System of Linear Equations

Moral

- Systems of linear equations can also be expressed using matrix notation.
- The determinant gives us a test for whether a system has a unique solution or not.
- Non-zero determinant allows us to easily find the unique solution of the system.
- A zero determinant tells us the system either has no solutions or it has infinitely many solutions.

System of Linear Equations

Moral

- Systems of linear equations can also be expressed using matrix notation.
- The determinant gives us a test for whether a system has a unique solution or not.
- Non-zero determinant allows us to easily find the unique solution of the system.
- A zero determinant tells us the system either has no solutions or it has infinitely many solutions.

System of Linear Equations

Moral

- Systems of linear equations can also be expressed using matrix notation.
- The determinant gives us a test for whether a system has a unique solution or not.
- Non-zero determinant allows us to easily find the unique solution of the system.
- A zero determinant tells us the system either has no solutions or it has infinitely many solutions.