First Order ODEs, Part II

Craig J. Sutton
craig.j.sutton@dartmouth.edu

Department of Mathematics
Dartmouth College

Math 23 Differential Equations
Winter 2013

Outline

Outline

1. Existence & Uniqueness Theorems
   - First Order Linear ODEs
   - General First Order ODEs
   - Linear vs. Non-Linear
2. Modeling
3. Autonomous Equations

Recap

We recall that we have techniques for solving some special ODE's:

- First Order Linear
- Separable
- Exact

However, it would be nice if there was a more general way to know if there's a solution just by looking at the ODE.

Recap

We recall that we have techniques for solving some special ODE's:

- First Order Linear
- Separable
- Exact

However, it would be nice if there was a more general way to know if there's a solution just by looking at the ODE.

Existence & Uniqueness: First Order Linear

Recall the following result for first-order linear ODEs.

Theorem (Thm. 2.4.1)

Consider the first order linear differential equation

$$y' + p(t)y = g(t); \; y(t_0) = y_0. \tag{1.1}$$

If the functions $p(t)$ & $g(t)$ are continuous on an open interval $a < t < b$ containing t_0, then there is a unique solution $y = \phi(t)$ defined on $a < t < b$ satisfying Eq. 1.1.

Examples

Example

Consider the first-order linear IVP $y' + 3t^2 y = \ln(t), y(1) = 4.$

- $p(t) = 3t^2$ is continuos on the whole real line.
- $g(t) = \ln(t)$ is continuous on the interval $0 < t < +\infty$.
- Hence, the largest interval containing $t_0 = 1$ on which $p(t)$ and $g(t)$ are both continuous is

$$0 < t < +\infty.$$

- Therefore, by the theorem the IVP has a unique solution $y(t)$ defined on all of $0 < t < +\infty$

Examples

Example

Consider the first-order linear IVP
$y' + \frac{t^2}{(t-2)} y = \frac{1}{(t+3)(t-5)}, y(-1) = \pi.$

- $p(t) = \frac{t^2}{(t-2)}$ is defined and continuous for $t \neq 2$.
- $g(t) = \frac{1}{(t+3)(t-5)}$ is defined and continuous for $t \neq -3, 5$.
- Hence, the largest interval containing $t_0 = -1$ on which $p(t)$ and $g(t)$ are both continuous is

$$-3 < t < 2.$$

- Therefore, by the theorem the IVP has a unique solution $y(t)$ defined on all of $-3 < t < 2$.

Exercises

In each of the following IVPs, find the largest interval of solution guaranteed to exist by the Existence & Uniqueness Theorem for First-Order Linear ODEs.

1. $ty' + 2y = 4t^2,\ y(1) = 2.$
2. $y' + \frac{2}{(t-4)}y = \ln(-(t+3)(t-11)),\ y(\frac{1}{2}) = 10^{10}.$
3. $e^{(t-3)(t+7)}y' - 3t^3y = \cos(t^3 - 3t),\ y(-5) = 3.$

Existence & Uniqueness: General First Order ODEs

Theorem

Consider the first order IVP

$$y' = f(t, y);\ y(t_0) = y_0. \tag{1.2}$$

If the functions f & $\frac{\partial f}{\partial y}$ are continuous on some rectangle

$$\mathcal{R} = \{(t, y) : \alpha < t < \beta \text{ and } \gamma < y < \delta\}$$

containing (t_0, y_0), then on some interval $(t_0 - h, t_0 + h) \subset (\alpha, \beta)$ there is a unique solution $y = \phi(t)$ to Eq 1.2.

Remark

If f is continuous on a neighborhood of (t_0, y_0), then a solution exists, **but** it need not be unique.

Using the Existence and Uniqueness Theorem

Example

Consider the IVP $\frac{dy}{dt} = f(t, y) = \frac{3t^2+4t+2}{2yt+t^2}$, $y(1) = 1$.

- f and $\frac{\partial f}{\partial y}$ exist and are continuous away from the lines $t = 0$ and $y = -\frac{t}{2}$.
- Let $\mathcal{R}$ be any rectangle around $(t_0, y_0) = (1, 1)$ which avoids these lines (e.g., $\mathcal{R} = \{(t, y) : 0 < t < 2, 0 < y < 2\}$).
- Then f and $\frac{\partial f}{\partial y}$ are both continuous on $\mathcal{R}$.
- Therefore, by the Existence & Uniqueness Theorem there is a unique solution $y(t)$ to the IVP on some interval containing $t_0 = 1$.
- Do you know how to find this unique solution?

Using the Existence and Uniqueness Theorem

Example

Consider the IVP $\frac{dy}{dx} = g(x,y) = \frac{3x^2+4x+2}{2(y-1)}$, $y(0) = -1$.

- g and $\frac{\partial g}{\partial y}$ are defined and continuous everywhere except on the line $y = 1$.
- Since $(x_0, y_0) = (0, -1)$ does not sit on the line $y = 1$, we may draw a rectangle $\mathcal{R}$ around it on which g and $\frac{\partial g}{\partial y}$ are continuous.
- Therefore, by the theorem there is a unique solution $y(x)$ to the IVP on some interval containing 0.
- Do you know how to find this unique solution?

Using the Existence and Uniqueness Theorem

Example

Consider the IVP $\frac{dy}{dx} = g(x,y) = \frac{3x^2+4x+2}{2(y-1)}$, $y(0) = +1$.

- g and $\frac{\partial g}{\partial y}$ are defined and continuous everywhere except on the line $y = 1$.
- Since $(x_0, y_0) = (0, 1)$ sits on the line $y = 1$, we cannot draw a rectangle $\mathcal{R}$ around it on which h and $\frac{\partial h}{\partial y}$ are continuous.
- Therefore, the Existence & Uniqueness theorem does not apply.
- Can we still find a solution? If so, is it unique?

Using the Existence and Uniqueness Theorem

Example

Consider the IVP $y' = h(t, y) = y^{\frac{1}{3}},\ y(0) = 0$.

- h is continuous everywhere, but $\frac{\partial h}{\partial y}$ does not exist at $(t_0, y_0) = (0, 0)$.
- Therefore, the Existence & Uniqueness theorem does not apply.
- Can we still find a solution? If so, is it unique?
- Find all the values of (t_0, y_0) for which the corresponding IVP has a unique solution?

Using the Existence and Uniqueness Theorem

Example

Consider the IVP $y' = k(x, y) = y^2, \ y(0) = 1.$

- k and $\frac{\partial k}{\partial y}$ are continuous everywhere.
- Therefore, the Existence & Uniqueness theorem the IVP has a unique solution.
- How large is the interval on which this solution exists?

Exercises

Determine what (if anything) the Existence and Uniqueness Theorems say about solutions to the following IVPs

1. $y' = \frac{t^2+ty+3}{t^2+y^2},\ y(0) = -1.$
2. $y' = \cos(t\sqrt{|y|}),\ y(1) = -3.$
3. $y' + \ln(t)y = \sin(t^2 + 3),\ y(0) = 3.$
4. $y' - |y|t = 0,\ y(1) = 0.$
5. $y' = \cos(y),\ y(1) = \pi/2.$

First Order ODEs: Linear vs. Non-Linear

1. **General Solution**
   - Linear Eqs.: (under a mild condition) we can obtain exact solutions using integrating factor.
   - Non-linear Eqs.: methods might miss some valid solutions.
2. **Interval of Definition**
   - Linear Eqs.: look for discontinuities in $p(t)$ & $g(t)$.
   - Non-linear Eqs.: not so easy.
3. **Explicit vs. Implicit Solutions**
   - Linear Eqs.: get explicit solutions (if you can perform integral).
   - Non-linear Eqs.: usually get implicit solutions. In practice need numerical techniques (e.g., Euler's method).

First Order ODEs: Linear vs. Non-Linear

1. **General Solution**
   - Linear Eqs.: (under a mild condition) we can obtain exact solutions using integrating factor.
   - Non-linear Eqs.: methods might miss some valid solutions.
2. **Interval of Definition**
   - Linear Eqs.: look for discontinuities in $p(t)$ & $g(t)$.
   - Non-linear Eqs.: not so easy.
3. **Explicit vs. Implicit Solutions**
   - Linear Eqs.: get explicit solutions (if you can perform integral).
   - Non-linear Eqs.: usually get implicit solutions. In practice need numerical techniques (e.g., Euler's method).

First Order ODEs: Linear vs. Non-Linear

1. **General Solution**
   - Linear Eqs.: (under a mild condition) we can obtain exact solutions using integrating factor.
   - Non-linear Eqs.: methods might miss some valid solutions.
2. **Interval of Definition**
   - Linear Eqs.: look for discontinuities in $p(t)$ & $g(t)$.
   - Non-linear Eqs.: not so easy.
3. **Explicit vs. Implicit Solutions**
   - Linear Eqs.: get explicit solutions (if you can perform integral).
   - Non-linear Eqs.: usually get implicit solutions. In practice need numerical techniques (e.g., Euler's method).

Outline

The Three Steps of Mathematical Modeling

1. **Construction:** State problem & assumptions about process invovled. Translate into mathematics.
2. **Analysis:** Solve model explicitly and/or gain qualitative/quantitative information about solution.
3. **Real World vs. Model:** A model is only as good as the predictions it makes.

Salt in a Tank

Problem

Suppose a tank contains 100 gallons of fresh water. Then water containing $\frac{1}{2}$lb. of salt per gallon is poured into the tank at a rate of 2 gal./min. and the mixture leaves the tank at a rate of 2 gal./min. After 10 minutes the process is stopped and fresh water is pumped in at the rate of 2 gal./min., with the mixture leaving the tank at the same rate. Find the amount of salt in the tank at the end of an additional 10 minutes.

From Rags to Riches

Problem

Suppose a person with no initial capital invests k dollars per year at an annual rate of return r. Assume that investments are made continuously and that the return is compounded continuously.

1. Determine the sum $S(t)$ accumulated after t years;
2. If $r = 7.5\%$ determine the rate k so that after 40 years our investor has \$ 1 million.
3. Suppose our investor can afford to save \$ 2000 per month. Determine the interest rate needed in order to accrue \$ 1 million after 40 years.

From Rags to Riches: Some Background

- If you invest S_0 at an annual rate of r compounded m times per year, then

$$S(t) = S_0(1 + \frac{r}{m})^{mt},$$

is your balance after t years.

- One can show that

$$\lim_{m \to \infty} S_0(1 + \frac{r}{m})^{mt} = S_0 e^{rt}$$

- Hence, when we say we invest at an annual rate of r compounded continuously we have

$$S(t) = S_0 e^{rt}.$$

From Rags to Riches: Some Background

- As an ODE a continuous growth rate is expressed as

$$\frac{dS}{dt} = rS$$

  A first order linear ODE.

- Now, suppose that deposits, withdrawals, etc. take place at a constant rate k, then

$$\frac{dS}{dt} = rS + k$$

- The solution is of the form

$$S(t) = S_0 e^{rt} + \frac{k}{r}(e^{rt} - 1),$$

  where S_0 is the initial investment.

From Rags to Riches

Problem

Suppose a person with no initial capital invests k dollars per year at an annual rate of return r. Assume that investments are made continuously and that the return is compounded continuously.

1. Determine the sum $S(t)$ accumulated after t years;
2. If $r = 7.5\%$ determine the rate k so that after 40 years our investor has \$ 1 million.
3. Suppose our investor can afford to save \$ 2000 per month. Determine the interest rate needed in order to accrue \$ 1 million after 40 years.

Outline

Equilibrium Solutions

Definition

Let $y' = f(t, y)$ be a first order ODE. If α is such that $f(t, \alpha) = 0$ for all t, then

$$y(t) = \alpha$$

is called an **equilibrium solution** of the ODE.

Equilibrium Solutions

Example

Consider the ODE $y' = \cos(t)(y^2 + y - 6)(y^2 - 16)$. It has equilibrium solutions:

- $y(t) = -3$
- $y(t) = 2$
- $y(t) = -4$
- $y(t) = 4$

Equilibrium Solutions

Example

Consider the ODE $y' = y^2 + 2y - 8$. It has equilibrium solutions:

- $y(t) = -4$
- $y(t) = 2$

Notice in this case $f(t, y)$ only depends on y.

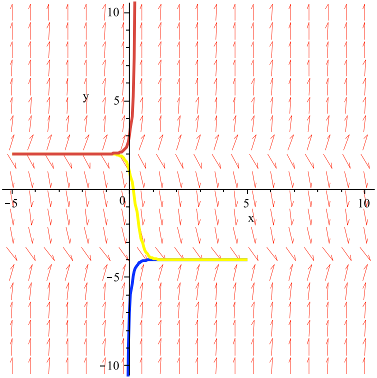
Equilibrium Solutions

Consider the ODE $y' = y^2 + 2y - 8$ and its equilibrium solutions: $y(t) = -4$ and $y(t) = 2$. Then

- $\frac{dy}{dt} > 0$ for $y > 2$;
- $\frac{dy}{dt} < 0$ for $-4 < y < 2$;
- $\frac{dy}{dt} > 0$ for $y < -4$;

We can see that $y(t) = 2$ is an unstable equilibrium while $y(t) = -4$ is a stable equilibrium...

Equilibrium Solutions



Equilibrium Solutions & Autonomy

Definition

A first order ODE of the form

$$y' = f(y)$$

is said to be **autonomous**. That is, the derivative of y with respect to t has no explicit dependence on t.

Equilibrium Solutions & Autonomy

The equilibrium solutions of $y' = f(y)$ correspond to the zeroes of $f(y)$.

Definition

Let $y(t) = K$ be an equilibrium solution of $y' = f(y)$, then:

1. ϕ is said to be an asymptotically stable solution if there is a δ such that if $y(t)$ is a solution to the IVP

$$y' = f(y), \; y(t_0) = y_0,$$

where $y_0 \in (K - \delta, K) \cup (K, K + \delta)$, then $\lim_{t\to\infty} y(t) = K$.