First Order ODEs, Part I

Craig J. Sutton
craig.j.sutton@dartmouth.edu

Department of Mathematics
Dartmouth College

Math 23 Differential Equations
Winter 2013

Outline

Outline

Outline

Outline

What is a First Order Linear Equation?

Definition

A general first-order linear ODE has the form

$$y' + p(t)y = g(t),$$

where it is understood that y is a function of t.

Example

A falling body of mass m is governed by the linear ODE

$$\frac{dv}{dt} = g - \frac{\gamma}{m}v.$$

An Idea

- Here is an “easy” first order equation:

$$y' = g(t).$$

- To solve it, we just integrate

$$y(t) = \int^{t} g(s)ds + C$$

- Can we reduce all first order linear ODEs to an (easy) integration problem?

Motivating Example

- Consider
$$y' + 7y = 3t \tag{1.1}$$
- Let $\mu(t) = e^{7t}$.
- Then $y(t)$ solves Eq 1.1 if and only if $y(t)$ solves
$$\mu(t)y' + \mu(t)7y = 3\mu(t)t \tag{1.2}$$
- But, using the product rule, we see the LHS of Eq 1.2 can be expressed as
$$\frac{d}{dt}(\mu(t)y).$$

Motivating Example

- Hence, $y(t)$ solves Eq 1.1 if and only if $y(t)$ satisfies

$$\frac{d}{dt}(e^{7t}y) = 3te^{7t} \tag{1.3}$$

- Integrating both sides we get:

$$y(t) = \frac{3}{7}t - \frac{3}{49} + Ce^{-7t}.$$

(Use initial conditions to solve for C.)

Motivating Example

Moral

We started with the equation

$$y' + 7y = 3t$$

and by multiplying this equation by $\mu(t) = e^{7t}$ we reduced our linear ODE to an easy integration problem. Consequently, we call $\mu(t)$ an **integrating factor**.

Motivating Example

How did we find the integrating factor $\mu(t) = e^{7t}$?

- Compare $\frac{d}{dt}(\mu(t)y)$ and $\mu(t)y' + 7\mu(t)y$.
- Equal if $\mu'(t) = 7\mu(t)$.
- $\mu(t) = e^{7t}$.

The Technique in General

- Let

$$y' + p(t)y = g(t) \tag{1.4}$$

be a general first order linear ODE, where p and g are **continuous.**

- For any $\mu(t) > 0$ we see $y(t)$ solves Eq 1.4 if and only if $y(t)$ solves

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t). \tag{1.5}$$

- Now, lets be clever about how we choose μ.

The Technique in General

- Let
$$y' + p(t)y = g(t) \tag{1.4}$$
be a general first order linear ODE, where p and g are **continuous.**
- For any $\mu(t) > 0$ we see $y(t)$ solves Eq 1.4 if and only if $y(t)$ solves
$$\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t). \tag{1.5}$$
- Now, lets be clever about how we choose μ.

The Technique in General

- Let
$$y' + p(t)y = g(t) \tag{1.4}$$
be a general first order linear ODE, where p and g are **continuous.**
- For any $\mu(t) > 0$ we see $y(t)$ solves Eq 1.4 if and only if $y(t)$ solves
$$\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t). \tag{1.5}$$
- Now, lets be clever about how we choose μ.

The Technique in General

- Let $\mu(t) = \exp(\int^t p(s)ds) > 0$.
- Then

$$y' + p(t)y = g(t) \Longleftrightarrow \frac{d}{dt}(\mu(t)y) = \mu(t)g(t). \qquad (1.6)$$

Why??

- Integrating Eq 1.6 we see

$$y(t) = \frac{\int_{t_0}^t \mu(s)g(s)ds + C}{\mu(t)} \qquad (1.7)$$

The Technique in General

- Let $\mu(t) = \exp(\int^t p(s)ds) > 0$.
- Then

$$y' + p(t)y = g(t) \iff \frac{d}{dt}(\mu(t)y) = \mu(t)g(t). \tag{1.6}$$

Why??

- Integrating Eq 1.6 we see

$$y(t) = \frac{\int_{t_0}^{t} \mu(s)g(s)ds + C}{\mu(t)} \tag{1.7}$$

The Technique in General

Definition

The function

$$\mu(t) = \exp(\int^t p(s)ds)$$

is called the **integrating factor** for Eq 1.4.

It allows us to substitute the (easy) integration problem

$$\frac{d}{dt}(\mu(t)y) = \mu(t)g(t)$$

for the linear ODE

$$y' + p(t)y = g(t).$$

The Technique in General

Definition

The function

$$\mu(t) = \exp\left(\int^t p(s)ds\right)$$

is called the **integrating factor** for Eq 1.4.
It allows us to substitute the (easy) integration problem

$$\frac{d}{dt}(\mu(t)y) = \mu(t)g(t)$$

for the linear ODE

$$y' + p(t)y = g(t).$$

The Technique in General

With a little bit of thought we can see that we've actually shown the following.

Theorem (2.4.1)

If p and g are cont. on an open interval $I = (\alpha, \beta)$ containing t_0, there is a unique function $y = \phi(t)$ on I that satisfies the IVP

$$y' + p(t)y = g(t), \; y(t_0) = y_0.$$

Exercises

1. Let $\mu(t) = \exp(\int^t p(s)ds)$. Check directly that $y(t) = \frac{\int^t \mu(s)g(s)ds + C}{\mu(t)}$ solves
$$y' + p(t)y = g(t).$$
2. Solve the initial value problem
$$y' - y = 2te^{2t}, \; y(0) = 1.$$
3. Solve the IVP
$$y' + \frac{2}{t}y = \frac{\cos(t)}{t}, \; y(\pi) = 0, t > 0.$$

Outline

First it will be useful to recall the method of integration by parts:

- Let $f(x)$ be an integrable function
- Let $g(x)$ be differentiable
- Then
$$\int f(g(x))g'(x)\,dx = \int f(u)\,du,$$
where $u = g(x)$.
- Or, recall that if $u = g(x)$ is differentiable, then
$$du = \frac{dg}{dx}\,dx = g'(x)\,dx$$
(do you remember differentials from calculus?)

- Recall that a general first order ODE is of the form

$$\frac{dy}{dx} = f(x, y)$$

- Such an equation can always be expressed as

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0.$$

(E.g., $M(x, y) = -f(x, y)$ and $N(x, y) = 1$.)

Definition

If a first order ODE $y' = f(x, y)$ can be expressed in the form

$$M(x) + N(y)y' = 0,$$

then we say the equation is **separable**.

Example

- Consider the non-linear ODE

$$\frac{dy}{dx} = \frac{x^3}{1-y}, \; y(0) = 1. \tag{2.1}$$

- This can be re-written as

$$-x^3 + (1-y)\frac{dy}{dx} = 0 \tag{2.2}$$

So, it is separable.

Example

We now observe...

- Rearranging we obtain

$$(1-y)\frac{dy}{dx} = x^3$$

- Integrating both sides w.r.t. x we obtain

$$\int (1-y)\frac{dy}{dx}\,dx = \int x^3\,dx.$$

- But, $dy = \frac{dy}{dx}\,dx$. (Why?).
- So we have

$$\int (1-y)\,dy = \frac{x^4}{4} + C.$$

Example

- Integrating we obtain

$$y - \frac{y^2}{2} = \frac{x^4}{4} + C.$$

which defines y implicitly as a function of x.

- Using our initial condition $y(0) = 1$ we get

$$C = -\frac{1}{2}.$$

- So y, the solution to our IVP, is defined implicitly by

$$y - \frac{y^2}{2} = \frac{x^4}{4} - \frac{1}{2}.$$

Example

Moral

The separability of our equation allowed us to reduce our work to an easy integration problem.

Question

Can we exploit separability in general?

The Technique

- Suppose we have a separable equation

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{2.3}$$

- Rearranging we get

$$N(y)\frac{dy}{dx} = -M(x) \tag{2.4}$$

- Let $y = y(x)$ be a differentiable function satisfying Eq. 2.4.
- Noticing $dy = \frac{dy}{dx}dx$ and integrating both sides of Equation 2.4 w.r.t. x we get

$$\int N(y)\,dy = -\int M(x)\,dx.$$

An implicit expression of y in terms of x.

The Technique

- Suppose we have a separable equation

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{2.3}$$

- Rearranging we get

$$N(y)\frac{dy}{dx} = -M(x) \tag{2.4}$$

- Let $y = y(x)$ be a differentiable function satisfying Eq. 2.4.
- Noticing $dy = \frac{dy}{dx}\,dx$ and integrating both sides of Equation 2.4 w.r.t. x we get

$$\int N(y)\,dy = -\int M(x)\,dx.$$

An implicit expression of y in terms of x.

The Technique

- Suppose we have a separable equation

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{2.3}$$

- Rearranging we get

$$N(y)\frac{dy}{dx} = -M(x) \tag{2.4}$$

- Let $y = y(x)$ be a differentiable function satisfying Eq. 2.4.
- Noticing $dy = \frac{dy}{dx}\,dx$ and integrating both sides of Equation 2.4 w.r.t. x we get

$$\int N(y)\,dy = -\int M(x)\,dx.$$

An implicit expression of y in terms of x.

The Technique

- Suppose we have a separable equation

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{2.3}$$

- Rearranging we get

$$N(y)\frac{dy}{dx} = -M(x) \tag{2.4}$$

- Let $y = y(x)$ be a differentiable function satisfying Eq. 2.4.
- Noticing $dy = \frac{dy}{dx}\,dx$ and integrating both sides of Equation 2.4 w.r.t. x we get

$$\int N(y)\,dy = -\int M(x)\,dx.$$

An implicit expression of y in terms of x.

The Technique

- So, we've seen that separability has led us to

$$\int N(y)\,dy = -\int M(x)\,dx,$$

  which (after integrating) implicitly defines y as function of x
- However, it's not always possible to explicitly solve the resulting expression for y as a function of x, although in theory we know such a function exists.
- In such cases one usually resorts to numerical methods to obtain an approximation of the exact solution.

Exercises

1. Solve the IVP
$$y' = \frac{1-2x}{y}, \ y(1) = -2.$$
2. Solve the differential equation $y' = \frac{3x^2-1}{3+2y}$.
3. For each value of α solve the IVP
$$\frac{dy}{dt} = y^2, \ y(0) = \alpha.$$
(What's the moral of this problem?)
4. Find all solutions to $xy' = (1-y^2)^{\frac{1}{2}}$.
(**Hint:** Do you remember how to compute $\frac{d}{dx}\sin^{-1}(x)$?)

Outline

As we noted earlier, any first order ODE

$$\frac{dy}{dx} = f(x, y)$$

can always be expressed as

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0.$$

Indeed, just take $M(x, y) = -f(x, y)$ and $N(x, y) = 1$
Now ...

Exact Differential Equation: The Definition

Definition

A first order ODE of the form

$$M(x,y) + N(x,y)y' = 0 \tag{3.1}$$

is said to be **exact** if there is a function $\Psi(x,y)$ such that

$$\frac{\partial \Psi}{\partial x} = M(x,y) \text{ and } \frac{\partial \Psi}{\partial y} = N(x,y).$$

What's so Special About Exact Equations?

- Suppose $M(x, y) + N(x, y)y' = 0$ is an exact equation.
- Let $\Psi(x, y)$ be as in the definition. Then we get

$$\begin{aligned} \frac{d}{dx}(\Psi(x,y)) &= \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y} y' \\ &= M(x,y) + N(x,y)y' \\ &= 0 \end{aligned}$$

- Integrating we obtain

$$\Psi(x, y) = C.$$

which implicitly defines y as a function of x.
- To determine C use initial condition $y_0 = y(x_0)$.

Example

- Consider the IVP

$$3x^2 - y + (2y - x)y' = 0, \; y(1) = 3.$$

- $\Psi(x, y) = x^3 - xy + y^2$ is such that $\frac{\partial \Psi}{\partial x} = 3x^2 - y$ and $\frac{\partial \Psi}{\partial y} = 2y - x$.
- Then our ODE becomes

$$\frac{d}{dx}\Psi(x, y) = 0.$$

- Integrating we get

$$x^3 - xy + y^2 = C.$$

- The initial condition $y(1) = 3$ then tells us

$$x^3 - xy + y^2 = 11.$$

Example

How did we find the function $\Psi(x,y)$?

- Since $\frac{\partial\Psi}{\partial x} = M(x,y) = 3x^2 - y$ integration shows

$$\begin{aligned}\Psi(x,y) &= \int M(x,y)dx + h(y) \\ &= x^3 - xy + h(y)\end{aligned}$$

- Then since $\frac{\partial}{\partial y}\Psi(x,y) = N(x,y) = 2y - x$ we see

$$h'(y) - x = 2y - x.$$

- Therefore $\Psi(x,y) = x^3 - xy + y^2$.
- Where have you used this procedure before?

Criteria for Exactness

Theorem

Let the functions $M(x,y), N(x,y), \frac{\partial M}{\partial y}$ *and* $\frac{\partial N}{\partial x}$ *be continuous in the rectangular region* $\mathcal{R} = [a,b] \times [c,d]$ *in the xy-plane. Then*

$$M(x,y) + N(x,y)y' = 0$$

is an exact equation in $\mathcal{R}$ *if and only if*

$$M_y(x,y) = N_x(x,y).$$

Notice that this applies to our previous example.

Exercise

Check whether each of the following is exact. If it is, then find the solution.

1. $(2x+3)+(2y-2)y'=0.$
2. $(3x^2-2xy+2)\,dx+(6y^2-x^2+3)\,dy=0$
3. $(2x+4y)+(2x-2y)y'=0$

Summary

In this module we have studied three types of first order equations:

- First order linear equations
- Separable equations
- Exact equations

What makes these equations special is that solving them essentially boils down to computing an *appropriate* integral.