Additional wave equation notes

Suppose you want solve

$$(***)\begin{cases} a^2 u_{xx} = u_{tt} & 0<x<L \quad t>0 \\ u(0,t) = u(L,t) = 0 & t \geq 0 \\ u(x,0) = f(x) & 0 \leq x \leq L \\ u_t(x,0) = g(x) & 0 \leq x \leq L. \end{cases}$$

We Don't know how to solve this, so lets try to make 2 problems we know how to solve.

let $u(x,t) = v(x,t) + w(x,t)$.

Choose $v(x,t)$ to be the solution of

$$(*)\begin{cases} a^2 v_{xx} = v_{tt} \\ v(0,t) = v(L,t) = 0 \\ v(x,0) = f(x) \\ v_t(x,0) = 0 \end{cases}$$

& Choose $w(x,t)$ to be the solution of

$$(**)\begin{cases} a^2 w_{xx} = w_{tt} \\ w(x,0) = 0 \\ w_t(x,0) = g(x) \\ w(0,t) = w(L,t) = 0 \end{cases}$$

We know how to solve both of these.

In class, we talked about how to solve ⊛. Now we will describe how to

Solve (∗∗)

let $W(x,t) = X(x)\,T(t)$.

Pluging into the PDE, we find.

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda$$

$\Rightarrow$ we get 2 DE.

① $X'' + \lambda X = 0$

② $T'' + a^2 \lambda T = 0$

The soln to ① $\Rightarrow X(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

BC tells us. $X(0) = c_1 = 0$.

$X(L) = c_2 \sin(\sqrt{\lambda}\, L) = 0 \Rightarrow \sqrt{\lambda}\, L = n\pi \quad n = 1, 2, \ldots$

$$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$$

Solns to ② are of the form.

$$T(t) = k_1 \cos\left(\frac{an\pi}{L} t\right) + k_2 \sin\left(\frac{an\pi}{L} t\right)$$

we know $W(x,0) = X(x)\,T(0) = 0 \Rightarrow T(0) = 0$

$T(0) = k_1 = 0$

$\Rightarrow T_n(t) = k_n \sin\left(\frac{an\pi}{L} t\right)$

$$\Rightarrow W(x,t) = \sum_{n=1}^{\infty} k_n \sin\left(\frac{an\pi t}{L}\right) \sin\left(\frac{n\pi}{L} x\right)$$

Now to find the coefficients k_n.

$$W_t(x,t) = \sum_{n=1}^{\infty} k_n \sin\left(\frac{n\pi x}{L}\right)\left(\frac{an\pi}{L}\right)\cos\left(\frac{an\pi t}{L}\right)$$

$$W_t(x,0) = \sum_{n=1}^{\infty} k_n \sin\left(\frac{n\pi x}{L}\right)\left(\frac{an\pi}{L}\right) = g(x).$$

Since this is an odd fourier series, we know

$$\frac{an\pi}{L} k_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\rightarrow k_n = \frac{2}{an\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

So the solution to (***) is

$$U(x,t) = V(x,t) + W(x,t)$$

$$= \sum_{n=1}^{\infty} c_n \cos\left(\frac{an\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1} k_n \sin\left(\frac{an\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

where $c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$$k_n = \frac{2}{an\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$