Ex Find a series solution to

$$y'' - xy' - y = 0$$

centered around $x_0 = 1$.

Soln We seek a solution $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$

Now we plug this into the DE to find a_n.

We need $y'(x) = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

Plug into DE.

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}}_{y''} \underbrace{- x \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}}_{-x\,y'} \underbrace{- \sum_{n=0}^{\infty} a_n (x-1)^n}_{-y} = 0.$$

First we need to address this. (ie. add 0)

NOTE: $x = (x-1)+1$

Expression now reads.

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - (x-1) \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=1}^{\infty} a_n (x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

Multiplying in the $(x-1)$, we get

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}}_{\substack{m=n-2 \\ n=m+2}} - \underbrace{\sum_{n=1}^{\infty} n a_n (x-1)^n}_{m=n} - \underbrace{\sum_{n=1}^{\infty} a_n (x-1)^{n-1}}_{\substack{m=n-1 \\ n=m+1}} - \underbrace{\sum_{n=0}^{\infty} a_n (x-1)^n}_{m=n} = 0$$

Fix exponents so they match.

Doing the subsitutions, we get the following expression.

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}(x-1)^m - \sum_{m=1}^{\infty} m a_m (x-1)^m - \sum_{m=0}^{\infty} a_{m+1}(x-1)^m - \sum_{m=0}^{\infty} a_m (x-1)^m = 0$$

This one starts at 1. Make others start at m=1

Taking out zero terms.

$$2a_2 - a_0 - a_1 + \sum_{m=1}^{\infty}(m+2)(m+1)a_{m+2}(x-1)^m - \sum_{m=1}^{\infty} a_m m (x-1)^m - \sum_{m=1}^{\infty} a_{m+1}(m+1)(x-1)^m$$

$$- \sum_{m=1}^{\infty} a_m (x-1)^m = 0.$$

Rewriting as one series, we get.

$$2a_2 - a_0 - a_1 + \sum_{m=1}^{\infty}\left[(m+2)(m+1)a_{m+2} - a_m m - a_{m+1}(m+1) - a_m\right](x-1)^m = 0$$

$$\Rightarrow 2a_2 - a_0 - a_1 = 0 \quad \Rightarrow \quad a_2 = \frac{a_0 + a_1}{2}$$

$$\Rightarrow (m+2)(m+1)a_{m+2} - (m+1)\left[a_{m+1} + a_m\right] = 0$$

$$\Rightarrow a_{m+2} = \frac{a_{m+1} + a_m}{m+2} \leftarrow$$ This is the recurrence Relation.

Lets get the 1st 4 non-zero terms in the series

Soln. Recall $a_2 = \frac{a_0 + a_1}{2}$

m	a_{m+2}
1	$a_3 = \frac{a_2 + a_1}{3} = \frac{a_2}{3} + \frac{a_1}{3} = \frac{a_0}{6} + \frac{a_1}{6} + \frac{a_1}{3} = \frac{a_0}{6} + \frac{a_1}{2}$
2	$a_4 = \frac{a_3 + a_2}{4} = \frac{1}{4}\left(\frac{a_0}{6} + \frac{a_1}{2}\right) + \frac{1}{4}\left(\frac{a_0}{2} + \frac{a_1}{2}\right)$
	$= a_0\left(\frac{1}{24} + \frac{1}{8}\right) + a_1\left(\frac{1}{4}\right)$

$$\Rightarrow y(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + \dots$$

$$= a_0 + a_1(x-1) + \left(\frac{a_0}{2} + \frac{a_1}{2}\right)(x-1)^2 + \left(\frac{a_0}{6} + \frac{a_1}{2}\right)(x-1)^3 + \dots$$

To find the 2 fundamental solutions group by a_0 & a_1

$$y(x) = a_0\left(1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \dots\right)$$

$$+ a_1\left((x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \dots\right)$$

Alternative way:

1^{st} let $a_1 = 0$. To get a_0 series (ie $y_1(x)$)

$$\rightarrow a_2 = \frac{a_0}{2}$$

m	a_{m+2}
1	$a_3 = \frac{a_2 + a_1}{3} = \frac{a_0}{6}$
2	$a_4 = \frac{a_3 + a_2}{4} = \frac{a_0}{6(4)} + \frac{a_0}{8}$

$$y_1(x) = 1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \left(\frac{1}{24} + \frac{1}{8}\right)(x-1)^4 + \cdots$$

2^{nd} let $a_0 = 0$ To get a_1 series (ie. $y_2(x)$)

$$\rightarrow a_2 = \frac{a_1}{2}$$

m	a_{m+2}
1	$a_3 = \frac{a_2 + a_1}{3} = \frac{a_1}{6} + \frac{a_1}{3} = \frac{a_1}{2}$
2	$a_4 = \frac{a_3 + a_2}{4} = \frac{1}{4}\left(\frac{a_1}{2}\right) + \frac{1}{4}\left(\frac{a_1}{2}\right) = \frac{1}{4}a_1$

$$\rightarrow y_2(x) = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \cdots$$

Section 5.3

Suppose we would like to solve the IVP

$$\begin{cases} y'' + p(x)y' + q(x)y = 0 . \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \end{cases}$$

We seek a series soln of the form.

$y(x) = \sum_{n=0}^{\infty} a_n (x - x_c)^n$ where x_c is the

$= a_0 + a_1(x - x_c) + a_2(x - x_c)^2 + \ldots$

center to be determined.

Plugging in the intial information we see

$$y(x_0) = \sum_{n=0}^{\infty} a_n (x_0 - x_c)^n = a_0 + a_1(x_0 - x_c) + a_2(x_0 - x_c)^2 + \ldots$$

$$= y_0$$

If $x_0 = x_c$ we are left w/ $a_0 = y_0$.

So choose $x_0 = x_c$.

$$\rightarrow y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n .$$

& $a_0 = y_0 = y(x_0)$

Now $y'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$

$= a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \dots$

$\rightarrow y'(x_0) = a_1 = y_1$

Now we know the power series we are finding is the Taylor series.

This means $a_n = \dfrac{y^{(n)}(x_0)}{n!}$

Note: So far we have

$a_0 = y(x_0) = y_0$

$a_1 = \dfrac{y'(x_0)}{1!} = y_1$

Now we need: $a_2 = \dfrac{y''(x_0)}{2!}$

By DE, $y''(x) = -[p(x)y'(x) + q(x)y(x)]$

$\rightarrow y''(x_0) = -[p(x_0)y'(x_0) + q(x_0)y(x_0)]$

Similarly, we can find $a_3 = \dfrac{y^{(3)}(x_0)}{3!}$

Since $y^{(3)}(x) = \dfrac{d}{dx}\left(y''(x)\right) = -[p(x)y''(x) + p'(x)y'(x) + q(x)y'(x) + q'(x)y(x)]$.

we can continue in this fashion.

lets do a specific example.

Ex Find The 1^{st} 3 non zero terms in the series Soln for

$$\begin{cases} y'' + xy' + y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

Choose center at $x_0 = 0$.

We know $a_0 = y(0) = 1$

$$a_1 = \frac{y'(0)}{1!} = 0$$

$$a_2 = \frac{y''(0)}{2!}$$

From DE, $y''(x) = -[x\,y'(x) + y(x)]$

$\Rightarrow y''(x_0) = -[0\,y'(0) + y(0)] = -1$

$\Rightarrow a_2 = -\frac{1}{2}$

$$a_3 = \frac{y^{(3)}(x_0)}{3!}$$

$$y^{(3)}(x) = \frac{d}{dx}\left(y''(x)\right) = -\left[xy''(x) + y'(x) + y'(x)\right]$$

$$= -\left[x y'' + 2y'(x)\right]$$

$y^3(0) = 0$

$\Rightarrow a_3 = 0$

$$a_4 = \frac{y^{(4)}(x_0)}{4!}$$

$$y^{4}(x) = \frac{d}{dx}\left(y^{(3)}(x)\right) = -\left[x\, y'''(x) + y''(x) + 2y''(x)\right]$$

$$= -\left[x\, y^{(3)}(x) + 3y''(x)\right]$$

$$\rightarrow y^{(4)}(0) = -3y''(0) = 3$$

$$\rightarrow a_4 = \frac{3}{4!}$$

$$\rightarrow y(x) = 1 - \frac{1}{2}x^2 + \frac{3}{4!}x^4 + \cdots$$

How do we know where the series is going to converge?

Ans: we can look at the radius of convergence of the series expansions of $p(x)$ & $q(x)$ to find a min radius of convergence.

Thm 5.3.1 If x_0 is an ordinary point of $P(x)y'' + Q(x)y' + R(x)y = 0$ $\left(p = \frac{Q}{P}\ \ q = \frac{R}{P}\right)$

Then the general soln is

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 y_1(x) + a_1 y_2(x)$$

where $y_1(x)$ & $y_2(x)$ are the fundamental solns.

Also, the radius of convergence for each of the series solns y_1 & y_2 is at least as large as the minimum radius of convergence of the series for p & q.

Ex Consider the DE

$$y'' + \frac{1}{1-x} y' + \frac{1}{1-2x} y = 0$$

What is the minimum radius of convergence?

We know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ converges for $|x| < 1 = \rho_1$

& $\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n$ converges for $|2x| < 1$
$\Rightarrow \rho_2 = 1/2$

$\Rightarrow$ radius of convergence of solution is at least
$\min\{1/2, 1\} = 1/2$.

Additional, examples in text.