Math 23 Spring 2012

Differential Equations

Second Midterm Exam

Wednesday May 16, 4:00-6:00 PM

Your name (please print): ______________________________

Instructor (circle one): Gillman, Gordon.

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You must justify your answers to receive full credit.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Please sign below if you would like your exam to be returned to you in class. By signing, you acknowledge that you are aware of the possibility that your grade may be visible to other students.

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For grader use only:

Problem	Points	Score
1	36	
2	18	
3	11	
4	15	
5	15	
6	5	
Total	100	

1. (36 points) For each of the following, you are given the eigenvalues and the eigenvectors of a 2×2 matrix A. You are to:

   - Write down the general solution of the system $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$.
   - Indicate which of the phase portraits shown below is the correct phase portrait of the system.

   (a) A has eigenvalues 3 and -1 and corresponding eigenvectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$.

   General solution:

   $$\bar{X}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-t}$$

   Phase portrait: (ii)

   (b) A has eigenvalues 3 and 1 and corresponding eigenvectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$.

   General solution:

   $$\bar{X}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{t}$$

   Phase portrait: (iii)

(c) A has eigenvalues $2i$ and $-2i$ and corresponding eigenvectors $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$.

General solution:

$$\begin{pmatrix} 1+i \\ 1 \end{pmatrix} [\cos 2t + i \sin 2t)$$

$$= \overset{\text{Real}}{\begin{pmatrix} \cos 2t - \sin 2t \\ \cos 2t \end{pmatrix}} + i \overset{\text{imag}}{\begin{pmatrix} \sin 2t + \cos 2t \\ \sin 2t \end{pmatrix}}$$

$$\overline{X} = C_1 \begin{pmatrix} \cos 2t - \sin 2t \\ \cos 2t \end{pmatrix} + C_2 \begin{pmatrix} \sin 2t + \cos 2t \\ \sin 2t \end{pmatrix}$$

Phase portrait: (iv)

(d) A has eigenvalues $1+2i$ and $1-2i$ and corresponding eigenvectors $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$.

General solution:

$$e^t \begin{pmatrix} 1+i \\ 1 \end{pmatrix} (\cos 2t + i \sin 2t)$$

$$= e^t \left[\begin{pmatrix} \cos 2t - \sin 2t \\ \cos 2t \end{pmatrix} + i \begin{pmatrix} \cos 2t + \sin 2t \\ \sin 2t \end{pmatrix} \right]$$

$$\bar{X}(t) = C_1 e^t \begin{pmatrix} \cos 2t - \sin 2t \\ \cos 2t \end{pmatrix} + C_2 e^t \begin{pmatrix} \cos 2t + \sin 2t \\ \sin 2t \end{pmatrix}$$

Phase portrait: (i)

2. (15 + 3 points)

(a) Find the solution of the initial value problem

$$\mathbf{x}' = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} \mathbf{x} \qquad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

(Note: you should get an eigenvalue of algebraic multiplicity two. If not, please go back and redo your computation of the eigenvalues before proceeding with the rest of the problem.)

1st compute eigenvalues $\begin{vmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{vmatrix} = (-1-\lambda)(-3-\lambda)+1$

$= \lambda^2 + 3\lambda + \lambda + 3 + 1$

$= \lambda^2 + 4\lambda + 4 = (\lambda+2)^2 = 0$

$\to \lambda = 2$ is an eigenvalue w/ multiplicity 2.

Now eigenvector:

$$\left[\begin{array}{cc|c} -1-(-2) & -1 & 0 \\ 1 & -3-(-2) & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \to x_1 = x_2$$

$$\bar{\omega} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Now Generalized eigenvector γ i.e. $(A-\lambda I)\gamma = \bar{\omega}$

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & -1 & 1 \end{array}\right] \to \gamma_1 - \gamma_2 = 1 \qquad \text{let } \gamma_2 = 0 \Rightarrow \gamma_1 = 1$$

$$\to \bar{\gamma} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The solution is $\bar{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} \right]$

(b) What happens to the solution as $t \to \infty$?

$$\bar{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow c_1 = 1$$

$$c_2 = 3 - c_1 = 2$$

$$\to \bar{x}(t) = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t + 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^{-2t}$$

$$\lim_{t \to \infty} \bar{x}(t) = \bar{0}$$

3. (11 points)

(a) Find a real number α so that the three vectors given by

$$\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\1\\4\end{bmatrix}, \begin{bmatrix}1\\-4\\\alpha\end{bmatrix}$$

are linearly *dependent.*

$$\begin{vmatrix}1 & 2 & 1\\2 & 1 & -4\\3 & 4 & \alpha\end{vmatrix} = 1\begin{vmatrix}1 & -4\\4 & \alpha\end{vmatrix} - 2\begin{vmatrix}2 & -4\\3 & \alpha\end{vmatrix} + 1\begin{vmatrix}2 & 1\\3 & 4\end{vmatrix}$$

$$= \alpha + 16 - 2(2\alpha + 12) + 8 - 3 = \alpha - 4\alpha + 16 - 24 + 5$$

$$= -3\alpha + 16 - 24 + 5 = -3\alpha - 3 = 0 \text{ when } \alpha = -1$$

(b) Find a linear relation among the three vectors in part (a), where α is the real number that you found in part (a). (Note: if you need extra space, please use the back of the page or the extra paper and note where the additional work is located.)

Goal: Find c_1, c_2, c_3 st

$$c_1\begin{pmatrix}1\\2\\3\end{pmatrix} + c_2\begin{pmatrix}2\\1\\4\end{pmatrix} + c_3\begin{pmatrix}1\\-4\\\alpha\end{pmatrix} = \bar{0}.$$

$4 - 6$
$1 - 3$
$1 - 4 = -3$
$-4 - 2 = -6$
$4 - 6 = -2$
$-1 - 3 = -2$

$$\rightarrow \left[\begin{array}{ccc|c}1 & 2 & 1 & 0\\2 & 1 & -4 & 0\\3 & 4 & -1 & 0\end{array}\right] \xrightarrow[(3)=(3)-3(1)]{(2)=(2)-2(1)} \left[\begin{array}{ccc|c}1 & 2 & 1 & 0\\0 & -3 & -6 & 0\\0 & -2 & -4 & 0\end{array}\right]$$

$$\xrightarrow{(2) = -\frac{1}{3}(2)} \left[\begin{array}{ccc|c}1 & 2 & 1 & 0\\0 & 1 & 2 & 0\\0 & -2 & -4 & 0\end{array}\right] \xrightarrow{(3) = (3) + 2(2)} \left[\begin{array}{ccc|c}1 & 2 & 1 & 0\\0 & 1 & 2 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

$$\rightarrow c_2 = -2c_3 \qquad c_1 = -2c_2 - c_3 \qquad \text{let } c_3 = 1$$

$$\rightarrow c_2 = -2$$
$$c_1 = 4 - 1 = 3$$

$$\rightarrow 3\begin{pmatrix}1\\2\\3\end{pmatrix} - 2\begin{pmatrix}2\\1\\4\end{pmatrix} + \begin{pmatrix}1\\-4\\-1\end{pmatrix} = \bar{0}$$

4. (15 points) Find the first four terms (i.e., up through and including the x^3 term) in the general power series solution about $x_0 = 0$ for the differential equation:

$$y'' + x^2y' + y = 0.$$

(Your solution will involve two arbitrary constants.)

Soluti on is on next page

Guess the solution is

$$y(x) = \sum_{n=0}^{\infty} a_n x^n .$$

Plug into DE

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$m = n-2$, $n = m+2$ $m = n+1$, $n = m-1$ $m = n$

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m + \sum_{m=2}^{\infty} (m-1) a_{m-1} x^m + \sum_{m=0}^{\infty} a_m x^m = 0 .$$

Take out $m=0$, & $m=1$ terms from 1st & 3rd series

$$\underbrace{2 a_2 + a_0}_{m=0} + \underbrace{[3(2) a_3 + a_1] x}_{m=1} + \sum_{m=2}^{\infty} [(m+2)(m+1) a_{m+2} + (m-1) a_{m-1} + a_m] x^m = 0 .$$

$$\Rightarrow 2a_2 + a_0 = 0 \quad \Rightarrow \quad a_2 = \frac{-a_0}{2}$$

$$6 a_3 + a_1 = 0 \quad \Rightarrow \quad a_3 = \frac{-a_1}{6}$$

$$(m+2)(m+1)a_{m+2} + (m+1)a_{m+1} + a_m = 0$$

$$a_{m+2} = -\left[\frac{(m+1)a_{m+1} + a_m}{(m+2)(m+1)}\right]$$

need 1^{st} 4 terms.

$$y(x) = a_0 + a_1 x - \frac{a_0}{2}x^2 + \frac{-a_1}{6}x^3 + \cdots$$

5. (15 points) Find the terms up through degree four (i.e., up through and including a_4x^4) in a power series solution $y = \sum_{n=0}^{\infty} a_n x^n$ for the initial value problem:

$$y'' + \cos(x)y = 0, \quad y(0) = 1, \; y'(0) = 0.$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum n a_n x^{n-1} \qquad y'' = \sum n(n-1) a_n x^{n-2}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$$

$$(\cos x)(y) = \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots\right)$$

$$= a_0 + a_1 x + \left(a_2 - \tfrac{1}{2}a_0\right)x^2 + \dots$$

$$y'' + \cos(x)\, y$$

$$(2a_2 + 6a_3 x + 12a_4 x^2 + \dots) + a_0 + a_1 x + \left(a_2 - \tfrac{1}{2}a_0\right)x^2 + \dots$$

$$y(0) = 1 \Rightarrow a_0 = 1 \qquad y'(0) = 0 \Rightarrow a_1 = 0$$

$$(2a_2 + 6a_3 x + 12a_4 x^2 + \dots) + \left\{1 + \left(a_2 - \tfrac{1}{2}\right)x^2 + \dots\right\} = 0$$

$$2a_2 + 1 = 0 \qquad a_2 = -\frac{1}{2} \qquad 12a_4 + a_2 - \frac{1}{2} = 0$$

$$6a_3 + 0 = 0 \qquad a_3 = 0 \qquad 12a_4 - \frac{1}{2} - \frac{1}{2} = 0$$

$a_4 = \frac{1}{12}$

$y = 1 - \frac{1}{2} x^2 + \frac{1}{12} x^4 + \cdots$

<u>Method 2</u> $\quad y'' = (-\cos x)\, y$

$y''(0) = -1\, y(0) = -1$

$y''' = (\sin x)\, y - \cos x\, y'$

$y'''(0) = 0$

$y^{(4)} = (\cos x)\, y + (\sin x)\, y' + (\sin x)\, y'$
$\quad - (\cos x)\, y''$

$y^{(4)}(0) = 1 - y''(0) = 1 - (-1) = 2$

$a_0 = 1 \quad a_1 = 0 \quad a_2 = \frac{y''(0)}{2!} = -\frac{1}{2}$

$a_3 = 0 \quad a_4 = \frac{y^{(4)}(0)}{4!} = \frac{2}{24} = \frac{1}{12}$

6. (5 points) Determine a lower bound for the radius of convergence of power series solutions about the point $x_0 = 0$ for the differential equation

$$(1 + 4t^4)y'' + y' + y = 0.$$

Rewrite $y'' + \frac{1}{1+4t^4} y' + \frac{1}{1+4t^4} y = 0$

By Thm 5.3.1, the radius of convergence for the series solution is at least as large as the radius of convergence for the series representation of $\frac{1}{1+4t^4}$

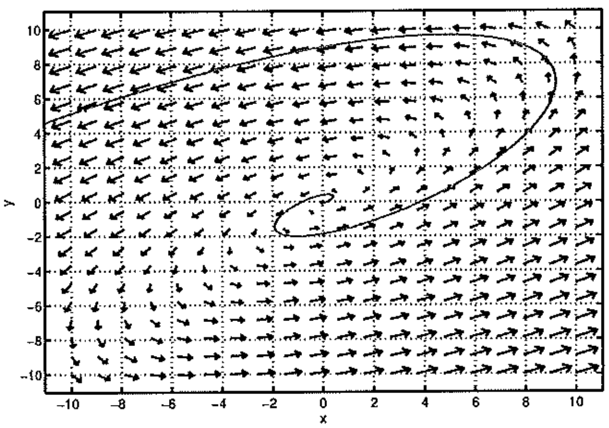
Well $\frac{1}{1+4t^4} = \frac{1}{1-(-4t^4)} = \sum_{n=0}^{\infty} (-4t^4)^n$ (Geometric series)

This converges for $|4t^4| < 1$

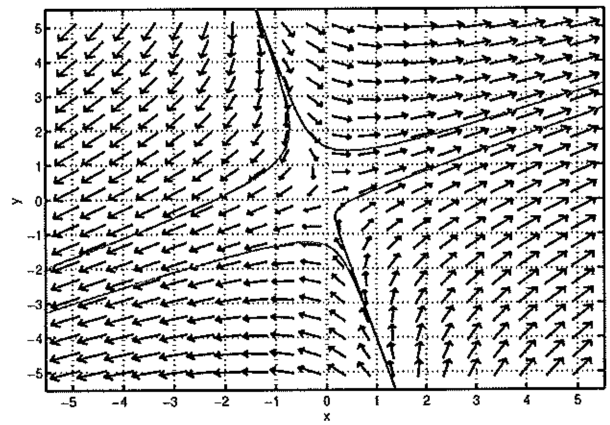
$\Rightarrow |t|^4 < 1/4$

$\Rightarrow |t| < (1/4)^{1/4}$

$\Rightarrow$ The min radius of convergence is $\rho = (1/4)^{1/4}$.



(i)



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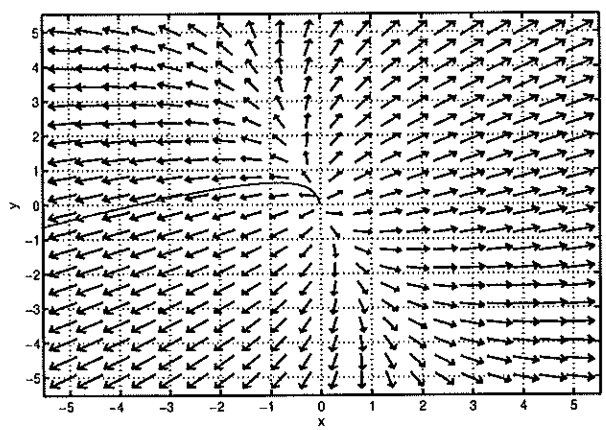

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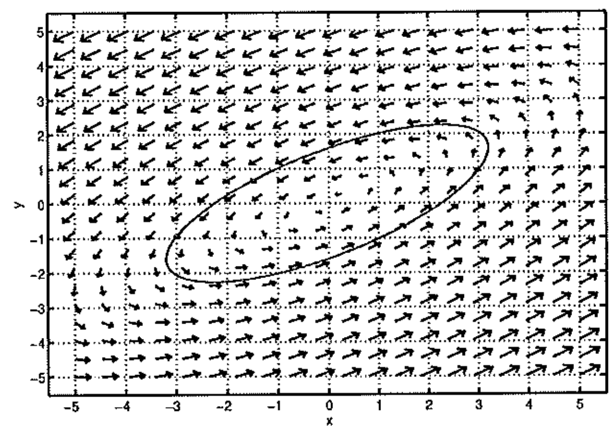

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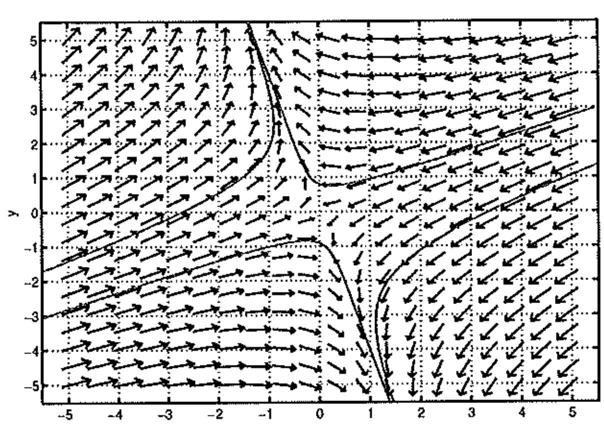

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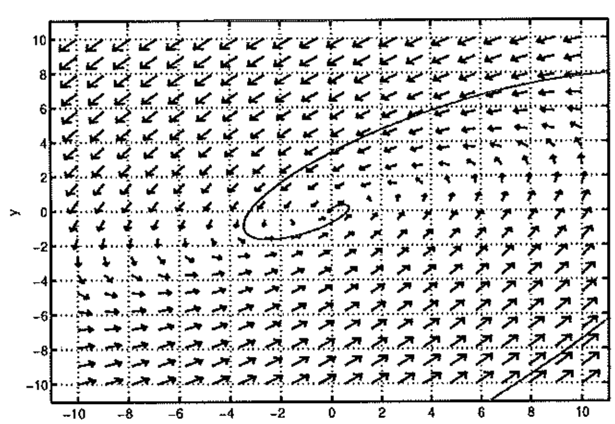

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