Worksheet #6

(1) Find the solution to

$$y'' + 8y' - 9y = 0 \qquad y(1) = 1, \qquad y'(1) = 0.$$

Describe its behavior as $t \to \infty$.

The characteristic eqn is $r^2 + 8r - 9 = 0$.

The roots can be found by factoring

$r^2 + 8r - 9 = (r+9)(r-1) = 0 \rightarrow r_1 = -9,\ r_2 = 1$

Soln is $y(t) = C_1 e^{-9t} + C_2 e^{t}$.

$y'(t) = -9C_1 e^{-9t} + C_2 e^{t}$

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(2) Determine the values of α for which the solution tends to zero as $t \to \infty$.

$$y'' + (3 - \alpha)y' - 2(\alpha - 1)y = 0$$

The characteristic eqn is $r^2 + (3-\alpha)r - 2(\alpha - 1) = 0$.

The roots are

$$r_{1,2} = \frac{-(3-\alpha) \pm \sqrt{(3-\alpha)^2 - 4(-2(\alpha-1))}}{2}$$

$$= \frac{-(3-\alpha) \pm \sqrt{9 - 6\alpha + \alpha^2 + 8\alpha - 8}}{2}$$

$$= \frac{1}{2}\left[-(3-\alpha) \pm \sqrt{\alpha^2 + 2\alpha + 1}\right]$$

(3) Determine the longest interval in which the given initial value problem is certain to have a unique twice differentiable solution.

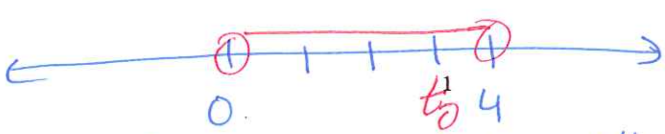
$$t(t-4)y'' + 3ty' + 4y = 2, \qquad y(3) = 0, \qquad y'(3) = -1$$

1st rewrite to make DE look like Thm.

$$y'' + \frac{3t}{t(t-4)}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$$

$p(t) = \frac{3t}{t(t-4)} \qquad q(t) = \frac{4}{t(t-4)} \qquad g(t) = \frac{2}{t(t-4)}$

$p(t)$, $q(t)$, & $g(t)$ are discontinuous at $t = 0$, & $t = 4$.



Interval for unique twice differentiable soln is $0 < t < 4$

1) continued

$$y(1) = c_1 e^{-9} + c_2 e = 1 \rightarrow c_1 = e^9(1 - e c_2)$$

$$y'(1) = -9c_1 e^{-9} + c_2 e = 0 \rightarrow c_2 = 9e^{-10}$$

$$\rightarrow c_1 = e^9 - e^{10}(9e^{-10}) = e^9 - 9$$

$$\rightarrow y(t) = (e^9 - 9)e^{-9t} + 9e^{-10}e^t$$

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \underbrace{c_1 e^{-9t}}_{\to 0} + \underbrace{c_2 e^t}_{\to\infty} \Rightarrow \infty.$$

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2 continued.

$$r_{1,2} = \frac{1}{2}\left[-(3-\alpha) \pm \sqrt{(\alpha+1)^2}\right] = \frac{1}{2}\left[-(3-\alpha) \pm (\alpha+1)\right]$$

$$r_1 = \frac{1}{2}(-3+\alpha+\alpha+1) = \frac{1}{2}(2\alpha - 2) = \alpha - 1$$

$$r_2 = \frac{1}{2}(-3+\alpha-\alpha-1) = \frac{1}{2}(-4) = -2$$

Soln is $y(t) = c_1 e^{(\alpha-1)t} + c_2 e^{-2t}$.

We want

$\lim_{t\to\infty} y(t) = 0.$ $\Rightarrow$ The exponents of both parts of solution must be negative

ie. $\alpha - 1 < 0 \rightarrow \alpha < 1$