Worksheet #21

(1) Consider the first order system

$$x' = \begin{bmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix} x$$

- Find the general solution and describe the behavior as $t \to \infty$.

1st find eigenvalues.

$$\begin{vmatrix} -3/2 - \lambda & 1 \\ -1/4 & -1/2 - \lambda \end{vmatrix} = (-3/2 - \lambda)(-1/2 - \lambda) + 1/4$$

$$= (3/4 + (3/2 + 1/2)\lambda + \lambda^2) + 1/4$$

$$= \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$$

$\rightarrow \lambda = -1$ is a repeated eigenvalue.

Now eigenvector

$$\left[\begin{array}{cc|c} -3/2+1 & 1 & 0 \\ -1/4 & -1/2+1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} -1/2 & 1 & 0 \\ -1/4 & 1/2 & 0 \end{array}\right] \xrightarrow[(2) = \frac{1}{2}(1) - (2)]{(1) = -2(1)} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\rightarrow x_1 = 2x_2$ let $x_2 = 1$ $\overline{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(*) - Draw a few trajectories.

other vector. $\overline{\gamma}$ (Generalized eigenvector) is the solution of $(A - \lambda I)\overline{\gamma} = \overline{w}$

$$\left[\begin{array}{cc|c} -1/2 & 1 & 2 \\ -\frac{1}{4} & 1/2 & 1 \end{array}\right] \xrightarrow[(2) = \frac{1}{2}(1) - (2)]{(1) = -2(1)} \left[\begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 0 & 0 \end{array}\right]$$

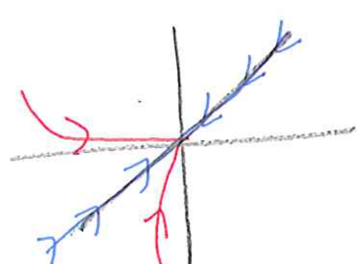
$\rightarrow \gamma_1 - 2\gamma_2 = -4$ let $\gamma_2 = 0$.

$\gamma_1 = -4$.

$\overline{\gamma} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}$

$\rightarrow$ General soln is $\overline{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2 e^{-t}\left[t\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -4 \\ 0 \end{pmatrix}\right]$

(*)



(2) Consider the first order system

$$x' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} x$$

- Find the general solution and describe the behavior as $t \to \infty$.

eigenvalues $\begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) + 8$

$= \lambda^2 + (-3+1)\lambda - 3 + 8$

$= \lambda^2 - 2\lambda + 5 = 0$

$\lambda = \dfrac{2 \pm \sqrt{4 - 4(5)}}{2} = 1 \pm 2i$

Eigenvectors for $\lambda = 1 + 2i$

$\left[\begin{array}{cc|c} 3-(1+2i) & -2 & 0 \\ 4 & -1-(1+2i) & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2-2i & -2 & 0 \\ 4 & -2-2i & 0 \end{array}\right]$

$① = \dfrac{2+2i}{8} ①$ $\left[\begin{array}{cc|c} 1 & -\frac{1}{4}(2+2i) & 0 \\ 4 & -2-2i & 0 \end{array}\right]$ $② = ② - 4①$ $\left[\begin{array}{cc|c} 1 & -1/4(2+2i) & 0 \\ 0 & 0 & 0 \end{array}\right]$

- Draw a few trajectories.

$x_1 = 1/4\,(2+2i)\,x_2$

let $x_2 = 4$

$\bar{w} = \begin{pmatrix} 2+2i \\ 4 \end{pmatrix}$

look at $\bar{w} e^{\lambda t} = \begin{pmatrix} 2+2i \\ 4 \end{pmatrix} e^{t} e^{2it}$

Collect real & imaginary parts.

$e^t \begin{pmatrix} 2+2i \\ 4 \end{pmatrix} (\cos 2t + i \sin 2t)$

$= e^t \left[\begin{pmatrix} 2\cos(2t) - 2\sin 2t \\ 4\cos(2t) \end{pmatrix} + i \begin{pmatrix} 2\cos(2t) + 2\sin(2t) \\ 4\sin 2t \end{pmatrix} \right]$

$x(t) = c_1 e^t \begin{pmatrix} 2\cos(2t) - 2\sin(2t) \\ 4\cos(2t) \end{pmatrix} + c_2 e^t \begin{pmatrix} 2\cos(2t) + 2\sin(2t) \\ 4\sin(2t) \end{pmatrix}$

Trajectories for #2.