Worksheet #15

(1) Expand to find the first 4 terms of the series

$$(\sin x)y$$

where $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

We know $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

To multiply series, collect like terms.

1	$0.$	x^3	$-\frac{a_0}{3!} + a_2$
x	a_0	x^4	$-\frac{a_1}{3!} + a_3$
x^2	a_1		

$\rightarrow (\sin x)y = a_0 x + a_1 x^2 + \left(-\frac{a_0}{3!} + a_2\right)x^3 + \left(a_3 - \frac{a_1}{3!}\right)x^4$

(2) Find the first 4 terms in the series solution of

$$y'' + (\sin x)y = 0.$$

let $x_0 = 0$. $\rightarrow y(x) = \sum_{n=0}^{\infty} a_n x^n$. $\rightarrow y''(x) = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$

Plugging into DE, we get

$0 = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + (\sin x)y = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + a_0 x + a_1 x^2 + \left(a_2 - \frac{a_0}{3!}\right)x^3 + \cdots$

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(3) Find the first 4 terms in the series solution of $y'' + xy' + y$ with initial conditions $y(2) = 0$ and $y'(2) = 1$.

Taylor series solutions are of the form $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$

where $a_n = \frac{y^{(n)}(x_0)}{n!}$

let $x_0 = 2$.

$\rightarrow y(2) = a_0 = 0.$

$y'(2) = a_1 = 1$

$a_2 = \frac{y^2(x_0)}{2!}$

$= -1$

$a_3 = \frac{y^{(3)}(x_0)}{3!}$

$= +2/3!$

$y''(x) = -[x y'(x) + y(x)]$

$y''(2) = -[2(1) + 0] = -2$

$y^{(3)}(x) = \frac{d}{dx}(y''(x))$

$= -[x y''(x) + y'(x) + y'(x)]$

$y^{(3)}(2) = -[2(-2) + 1 + 1] = +2$

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Problem2 continued

expanding both series we get

$$\underbrace{2a_2 + 6a_3x + 4(3)a_4x^2 + 5(4)a_5x^3 + 6(5)a_6x^4}_{y''}$$

$$+ \underbrace{a_0x + a_1x^2 + \left(a_2 - \frac{a_0}{3!}\right)x^3 + \left(a_3 - \frac{a_1}{3!}\right)x^4}_{(\sin x)y} + \cdots = 0$$

now collecting like terms.

$$\begin{array}{l|l}
1 & 2a_2 = 0 \rightarrow a_2 = 0 \\
x & 6a_3 + a_0 = 0 \rightarrow a_3 = -\frac{a_0}{6} \\
x^2 & 12a_4 + a_1 = 0 \rightarrow a_4 = \frac{-a_1}{12} \\
x^3 & 5(4)a_5 + a_2 - \frac{a_0}{3!} = 0 \rightarrow a_5 = \frac{a_0}{3!(5)(4)} = \frac{a_0}{5!} \\
x^4 & 30a_6 + a_3 - \frac{a_1}{3!} = 0 \rightarrow a_6 = \left(-a_3 + \frac{a_1}{3!}\right)\frac{1}{30}
\end{array}$$

$$= \left(-\frac{a_0}{6} + \frac{a_1}{3!}\right)\frac{1}{30}$$

$$\rightarrow y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots$$

$$= a_0 + a_1x + -\frac{a_0}{6}x^3 - \frac{a_1}{12}x^4 + \frac{a_0}{5!}x^5 + \left(\frac{-a_0}{6} + \frac{a_1}{3!}\right)\frac{1}{30}x^6 + \cdots$$

$$= a_0\underbrace{\left(1 - \frac{1}{6}x^3 + \frac{1}{5!}x^5 - \frac{1}{6}\left(\frac{1}{30}\right)x^6 + \cdots\right)}_{y_1(x)} + a_1\underbrace{\left(x - \frac{1}{12}x^4 + \frac{1}{3!}\left(\frac{1}{3}\right)x^6 + \cdots\right)}_{y_2(x)}$$

$$a_4 = \frac{y^4(x_0)}{4!}$$

$$y^4(x) = \frac{d}{dx}\left(y^{(3)}(x)\right) = \frac{d}{dx}\left(-\left[x\,y''(x) + 2y'(x)\right]\right)$$

$$= -\left[x\,y^{(3)}(x) + y''(x) + 2y''(x)\right]$$

$$y^4(x_0) = -\left[2(2) + -2 + 2(-2)\right] = 2$$

$$\Rightarrow a_4 = \frac{2}{4!}$$

$$\Rightarrow y(x) = a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3 + a_4(x-3)^4 + \cdots$$

$$= (x-2) + (-1)(x-2)^2 + 2/3!\,(x-2)^3 + \frac{2}{4!}(x-3)^4 + \cdots$$