Worksheet #13

(1) Find the interval and radius of convergence for the power series.

$$\sum_{n=0}^{\infty}(-1)^n \frac{(x-3)^n}{2n+1}$$

Use ratio test. we want $\lim_{n\to\infty} \left| \frac{a_{n+1}(x-3)^{n+1}}{a_n (x-3)^n} \right| < 1$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-3)^{n+1}}{\underbrace{2(n+1)+1}_{2n+3}} \cdot \frac{2n+1}{(-1)^n (x-3)^n} \right| = \lim_{n\to\infty} |x-3| \underbrace{\left(\frac{2n+1}{2n+3}\right)}_{\to 1} < 1$$

$\Rightarrow |x-3| < 1$ $\Rightarrow$ Radius of convergence is $\boxed{\rho = 1}$

Next page.

(2) Find the Taylor series for $f(x) = \ln x$ centered about $x_0 = 1$. What is the radius of convergence for the series? NOTE! There are 2 ways to do this problem.

Way 1: we know $f(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$ where $a_n = \frac{f^{(n)}(x_0)}{n!}$

Now, $f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2}$

$f^{(3)}(x) = \frac{(-1)^2 2}{x^3}$

$f^{(4)}(x) = \frac{(-1)^3 3\cdot 2\cdot 1}{x^4}$

$\to$ $f^{0}(x_0) = \ln 1 = 0$

$f^{(n)}(x_0) = \frac{(-1)^{n-1}}{x_0^{\,n}} = \frac{(-1)^{n-1}}{1^n} = (-1)^{n-1}$

$\to f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}(x-1)^n (n-1)! = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$

(3) Rewrite the series expression as a sum whose generic term involves x^n.

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n$$

1st multiply in the x.

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1}$$

now make both series have the same ~~terms~~ x^m

let $m = n-1$
$n = m+1$

let $m = n+1$
$n = m-1$

Now series can be written

$$\sum_{m=0}^{\infty} (m+1) a_{m+1} x^m + \sum_{m=1}^{\infty} a_{m-1} x^m$$

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Problem 1 continued

expanding to form interval. we get

$$-1 < x-3 < 1 \rightarrow 2 < x < 4$$

Now we must check endpoints.

at $x=2$. the series is.

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^n}{2n+1} = \sum_{n=0}^{\infty} \frac{1}{2n+1}$$ which is not

a convergent series. (Use integral test)

at $x=4$, the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$ This series converges by

alternating series test.

$\Rightarrow$ The interval of convergence is. $\boxed{2 < x \leq 4}$

Problem 2 continued

Way 2:

We know $f(x) = \ln x = \int \frac{1}{x}\,dx$

Now $\frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{n=0}^{\infty} (1-x)^n$

Add zero $\qquad$ geometric series

$$\rightarrow f(x) = \int \sum_{n=0}^{\infty} (1-x)^n\,dx = \sum_{n=0}^{\infty} \int (1-x)^n\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)(1-x)^{n+1}}{(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)(-1)^{n+1}(x-1)^{n+1}}{(n+1)}$$

To get this series to look the same as way 1

let $m = n+1$

$n = m-1$

$$\rightarrow f(x) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}(x-1)^m}{m}$$

Problem 3 continued.

Since indices do not match we must remove the zero $(m=0)$ term from the 1st series.

Thus we have.

$$(*) = (1)\,a_1 x^0 + \sum_{m=1}^{\infty} (m+1)\,a_{m+1}\,x^m + \sum_{m=1}^{\infty} a_{m-1}\,x^m$$

Since both series have same indexing & same powers of x. we can simplify by grouping like terms.

$$(*) = a_1 + \sum_{m=1}^{\infty} \left[(m+1)\,a_{m+1} + a_{m-1}\right] x^m$$