Homework #23

1) §10.5 #3

Assume $u(x,t)$ is separable. ie, $u(x,t) = X(x)T(t)$.
Plug into PDE. If we can make 2 independent
DE then it is separable.

$$[p(x)u_x]_x - r(x)u_{tt} = 0$$

$$\Rightarrow p(x)u_{xx} + p'(x)u_x - r(x)u_{tt} = 0$$

Plugging $u(x,t) = X(x)T(t)$ into PDE we get

$$p(x)X''T + p'(x)X'T - r(x)XT'' = 0$$

$$\Rightarrow (p(x)X'' + p'(x)X')T = r(x)XT''$$

$$\Rightarrow \frac{p(x)X'' + p'(x)X'}{r(x)X} = \frac{T''}{T} = -\lambda \qquad \lambda \text{ constant.}$$

$\Rightarrow$ separable.

The DE are

$$T'' + \lambda T = 0 \quad \& \quad p(x)X'' + p'(x)X' + \lambda r(x)X(x) = 0.$$

2) §10.5 # 5

Guess $U(x,y) = X(x)Y(y)$. Plug into PDE.

$$U_{xx} + (x+y)U_{yy} = 0$$

$$\rightarrow X''Y + (x+y)XY'' = 0$$

$$X''Y + xXY'' = -yY''$$

impossible to separate $\Rightarrow$ not separable

3) §10.5 #6

Guess $U(x,y) = X(x)Y(y)$. Plug into PDE.

$$U_{xx} + U_{yy} + xU = 0$$

$$\rightarrow X''Y + XY'' + xXY = 0$$

Move Y''

$$\rightarrow (X'' + xX)Y = -Y''X$$

Divide

$$\rightarrow \frac{(X'' + xX)}{X} = \frac{-Y''}{Y} = -\lambda$$

$\rightarrow$ 2 DE's are

$$X'' + xX + \lambda X = 0$$

$$-Y'' + \lambda Y = 0.$$

#4 §10.5 #8

Goal solve

$$u_{xx} = 4u_t \qquad 0 < x < 2 \qquad t > 0$$

$$u(0,t) = 0 \qquad u(2,t) = 0 \qquad t > 0.$$

$$u(x,0) = 2\sin\left(\frac{\pi x}{2}\right) - \sin(\pi x) + 4\sin(2\pi x) = f(x)$$

Soln

$$u(x,t) = X(x)T(t)$$

Plug in & separate

$$\frac{X''}{X} = \frac{4T'}{T} = -\lambda$$

$$\rightarrow \quad X'' + \lambda X = 0$$

$$\frac{T'}{T} = \frac{-\lambda}{4} \quad \rightarrow T(t) = Ce^{-\frac{\lambda}{4}t}$$

$$X(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$$

$$X(0) = 0 = c_1$$

$$X(2) = c_2\sin(2\sqrt{\lambda}) = 0$$

$$\rightarrow 2\sqrt{\lambda} = n\pi \qquad n = 1, 2, \ldots$$

$$\rightarrow \lambda = \left(\frac{n\pi}{2}\right)^2$$

$$\rightarrow X_n = \sin\left(\frac{n\pi}{2}x\right)$$

$$u_n(x,t) = C_n e^{-\left(\frac{n\pi}{2}\right)^2\frac{t}{4}}\sin\left(\frac{n\pi}{2}x\right)$$

$$U(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{2}\right)^2 \frac{t}{4}} \sin\left(\frac{n\pi x}{2}\right)$$

$$U(x,0) = \sum_{n=0}^{\infty} c_n \sin\left(\frac{n\pi x}{2}\right) = f(x)$$

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \int_0^2 \left(2\sin\left(\frac{\pi x}{2}\right) - \sin(\pi x) + 4\sin(2\pi x)\right)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \begin{cases} 2 & \text{if } n=1 \\ -1 & \text{if } n=2 \\ 4 & \text{if } n=4 \\ 0 & \text{otherwise} \end{cases}$$

By othogonality of sine functions.

$$\Rightarrow U(x,t) = 2\sin\left(\frac{\pi x}{2}\right) e^{-\left(\frac{\pi}{2}\right)^2 \frac{t}{4}} - \sin(\pi x)\, e^{-\frac{\pi^2}{4} t} + 4\sin(2\pi x)\, e^{-\left(\frac{4\pi}{2}\right)^2 \frac{t}{4}}$$

#5 § 10.5

9) $L = 40$ $\qquad\qquad\qquad\qquad u_t = \alpha^2 u_{xx}$

$\alpha^2 = 1$

$u(0, t) = u(L, t) = 0.$

$u(x, 0) = 50 \qquad 0 < x < 40.$

By previous problem, soln is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{40}\right)^2 t} \sin\left(\frac{n\pi x}{40}\right)$$

$$u(x, 0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{40}\right) = 50.$$

$$\Rightarrow c_n = \frac{2}{40}\int_0^{40} 50 \sin\left(\frac{n\pi x}{40}\right) dx$$

$$= \frac{50}{20} \frac{-40}{n\pi} \cos\left(\frac{n\pi x}{40}\right)\Big|_0^{40}$$

$$= -\frac{100}{n\pi}\Big(\underbrace{\cos(n\pi)}_{(-1)^n} - 1\Big)$$

$$= -\frac{100}{n\pi}\left((-1)^n - 1\right) = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{200}{n\pi} & \text{if } n \text{ odd} \end{cases}$$

or $\frac{100}{n\pi}\left(1 - \cos(n\pi)\right)$

$$\Rightarrow u(x,t) = \frac{100}{\pi}\sum_{n=1}^{\infty} \frac{(1-\cos(n\pi))}{n} e^{-\left(\frac{n\pi}{40}\right)^2 t} \sin\left(\frac{n\pi x}{40}\right)$$