2. We find the eigenvalues:

$$\begin{vmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 5$$

so $\lambda = -1 \pm 2i$

subsituting in $\lambda = -1-2i$ we get

$$\begin{pmatrix} -1+1+2i & -4 & 0 \\ 1 & -1+1+2i & 0 \end{pmatrix} \to \begin{matrix} 1 & 2i & 0 \\ 0 & 0 & 0 \end{matrix}$$

So our eigenvector is $\begin{pmatrix} -2i \\ 1 \end{pmatrix}$

A solution is then $\begin{pmatrix} -2i \\ 1 \end{pmatrix} e^{(-1-2i)t}$

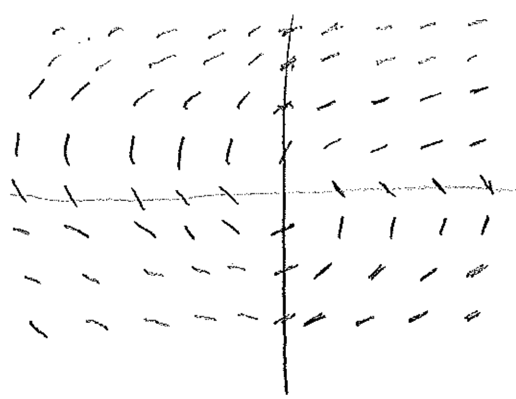
$$= e^{-t}\begin{pmatrix} -2i \\ 1 \end{pmatrix}(\cos(2t) - i\sin(2t))$$

$$= e^{-t}\begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix} + i e^{-t}\begin{pmatrix} -2\cos(2t) \\ \sin(2t) \end{pmatrix}$$

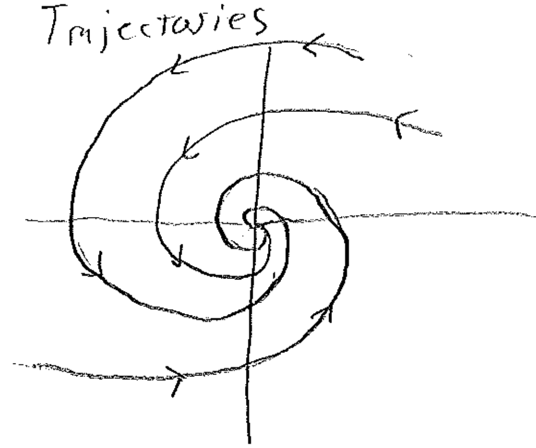
So, a general solution is given by

$$x = c_1 e^{-t}\begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -2\cos(2t) \\ \sin(2t) \end{pmatrix}$$

Direction field:



Trajectories



as $t \to \infty$ the solutions spiral in towards 0.

8. We find the eigenvalues:

$$\begin{vmatrix} -3-\lambda & 0 & 2 \\ 1 & -1-\lambda & 0 \\ -2 & -1 & -\lambda \end{vmatrix} = \lambda^3 + 4\lambda^2 + 7\lambda + 6$$

$$\Rightarrow \lambda = -2, \ -1 \pm \sqrt{2}\, i$$

For $\lambda = -2$ we find:

$$\left(\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & -1 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

so $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector.

For $\lambda = -1 - \sqrt{2}\, i$

$$\begin{pmatrix} -2+\sqrt{2}i & 0 & 2 & 0 \\ 1 & \sqrt{2}i & 0 & 0 \\ -2 & -1 & 1+\sqrt{2}i & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -2+\sqrt{2}i & 0 & 2 & 0 \\ 1 & \sqrt{2}i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so $\begin{pmatrix} -i\sqrt{2} \\ 1 \\ -1-i\sqrt{2} \end{pmatrix}$ is an eigen vector

So a solution is

$$e^{(-1-2i)t}\begin{pmatrix} -i\sqrt{2} \\ 1 \\ -1-i\sqrt{2} \end{pmatrix} = e^{-t}\begin{pmatrix} -\sqrt{2}\sin\sqrt{2}t \\ \cos\sqrt{2}t \\ -\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t \end{pmatrix} + i e^{-t}\begin{pmatrix} -\sqrt{2}\cos\sqrt{2}t \\ -\sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t - \sin\sqrt{2}t \end{pmatrix}$$

And the general solution is

$$x = c_1 e^{-2t}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -\sqrt{2}\sin\sqrt{2}t \\ -\cos\sqrt{2}t \\ -\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t \end{pmatrix} + c_3 e^{-t}\begin{pmatrix} -\sqrt{2}\cos\sqrt{2}t \\ -\sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t - \sin\sqrt{2}t \end{pmatrix}$$

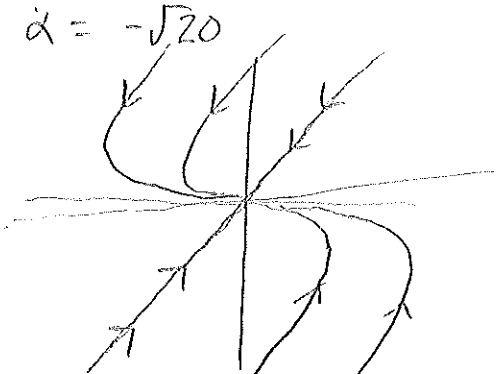
14. a. The eigenvalues are the roots of

$$\begin{vmatrix} -\lambda & -5 \\ 1 & \alpha-\lambda \end{vmatrix} = \lambda^2 - \alpha\lambda + 5 = 0$$

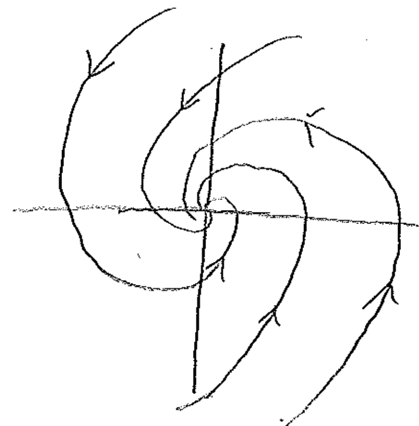
$$\lambda = \frac{\alpha}{2} \pm \frac{\sqrt{\alpha^2-20}}{2}$$

b. The system has critical values at $\alpha = \pm\sqrt{20}, 0$
Namely the points where the eigenvalues transition from real to complex, and where the real part of the eigenvalue changes sign.

c. $\alpha = -\sqrt{20}$

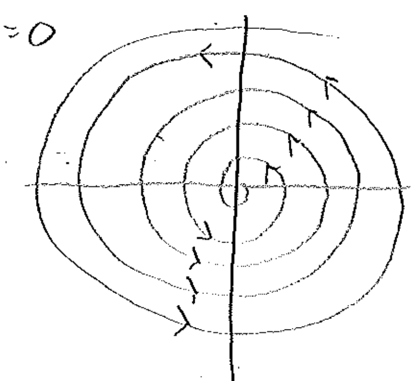


below



above

$\alpha = 0$



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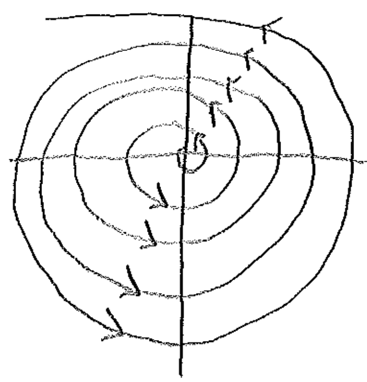
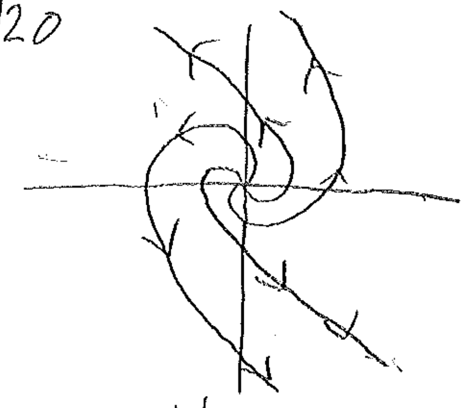


above

$\alpha = \sqrt{20}$



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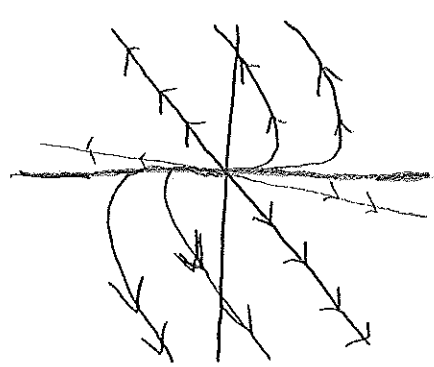


above

28. a. $x_1 = u$, $x_2 = u'$

$$x_2' = u''$$

so $m x_2' + k x_1 = 0 \Rightarrow x_2' = -\frac{k}{m} x_1$

and $x_1' = u' = x_2$

so $$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

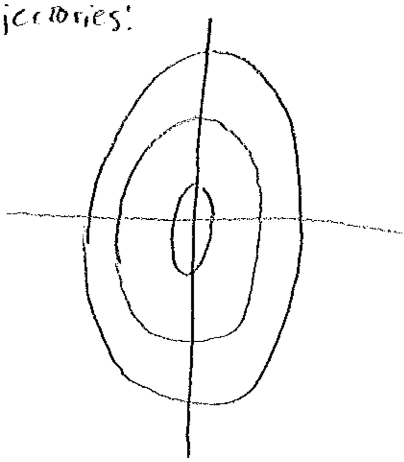
b. To find the eigenvalues:

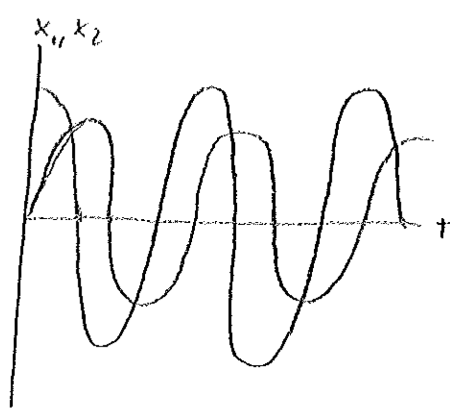
$$\begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\lambda \end{vmatrix} = \lambda^2 + \frac{k}{m}$$

so $\lambda = \pm i\sqrt{\frac{k}{m}}$

c. Since the eigenvalues are purely imaginary, the trajectories are ellipses or circles. Testing the point $(0,1)$ we see that this flow is clockwise.

Trajectories:





d. Since the imaginary part of the eigenvalue is $\sqrt{\frac{k}{m}}$ we see that the natural frequency will be $\omega_0 = \sqrt{\frac{k}{m}} = |\lambda|$.