Dartmouth College
Mathematics 23 - Assignment 20

1. Boyce and DiPrima Section 7.8: 2

2. Boyce and DiPrima Section 7.8: 6

3. Boyce and DiPrima Section 7.8: 7(a)

4. As you know, the initial value problem $y' = ay$, $y(0) = y_0$ has solution $e^{at}y_0$. Systems

$$\mathbf{x}' = A\mathbf{x}$$

where as usual A is a square matrix, can be solved in the same way. The exponential of a matrix B is defined by

$$e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} = \dots$$

where the powers of B are computed by matrix multiplication. One can then show that the solution to $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \xi$ is given by

$$\mathbf{x}(t) = e^{tA}\xi.$$

We do an example here. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

(a) Using induction, show that

$$(tA)^n = \begin{bmatrix} t^n & nt^n \\ 0 & t^n \end{bmatrix}$$

(b) Show that

$$e^{tA} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{t^n}{n!} & \sum_{n=0}^{\infty} \frac{nt^n}{n!} \\ 0 & \sum_{n=0}^{\infty} \frac{t^n}{n!} \end{bmatrix}$$

Each of the power series appearing in the matrix is a familiar function (e.g., two of them are e^t); figure out what the third one is.

(c) Given the initial condition $\mathbf{x}(0) = \begin{bmatrix} a \\ b \end{bmatrix}$, write down the solution $\mathbf{x} = e^{tA}\mathbf{x}_0$.

(d) Solve the system $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} a \\ b \end{bmatrix}$ in the usual way and compare your answer with part (c).

5. Solve the system

$$\mathbf{x}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

in two ways: the usual method and using the matrix exponential as in the previous problem. Compare your solutions. (Note: the entries of e^{tA} will be familiar power series.)